CARLETON UNIVERSITY

SCHOOL OF
MATHEMATICS AND STATISTICS

HONOURS PROJECT

TITLE: Characteristic Classes and de Rham Cohomology

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DATE: April 14, 2021

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1. Introduction

The classification problem of spaces is one of importance in geometry, but general topological spaces can behave in very weird ways. It is common to try to focus on a subclass of topological spaces and hope to reduce a geometric problem to an algebraic one by using tools from algebraic topology, such as homotopy groups or homology. A particular type of such a tool is de Rham cohomology, H^*, which associates to a space M, which has a differentiable structure, a graded algebra, $H^*(M)$, whose elements are, vaguely speaking, elements of the form $f \cdot dx^{i_1} \wedge \cdots \wedge dx^{i_k}$, where $f : M \to \mathbb{R}$ is a differentiable function. This is one step towards reducing a geometric problem to an algebraic one, however a differentiable space M can have too many differentiable functions to analyze, so it would be ideal to associate to M a single element $c(M)$ which belongs to $H^*(M)$, such that this element naturally encodes geometric properties of M. This is the idea of characteristic classes. One can argue that the concept was born at the time of Euler, when his formula $V - E + F = 2$ for a convex regular polyhedron with V vertices, E edges and F faces was discovered, but a proper development wasn't done until 1935 by Whitney and Stiefel. They found an invariant that represents an obstruction to the existence of vector fields on a space M that locally look like the product of $U \subset M$ with a sphere.

The aim of this paper is to introduce the reader to the general theory of spaces with differentiable structures and vector bundles over them, which are spaces that locally look like $U \times \mathbb{R}^k$. Sections 1 and 2 give the reader some basic definitions and results from the theory of smooth manifolds and vector bundles. These are mainly based on the books [1], [2] and [3]. The reader who has a basic understanding of these topics is encouraged to skip the first two sections and refer back to these for reference when needed. Section 4 and 5 are based on [4], where we discuss the de Rham cohomology of a space M, and show that the de Rham cohomology ring of M is the same as the de Rham cohomology of the vector bundle E over it. This relation is a fundamental step towards the classification problem, since it gives rise to many important invariants on M. These invariants, discussed in section 6 and based on the books [4], [5] and [3], are called characteristic classes and live in the cohomology ring of M. We conclude this exposition in section 7 with further information on the classification problem and where to go from there. We aim to give enough background for the reader to tackle more advanced problems in the classification of differentiable spaces. Although many of these advanced problems tend to be tackled in more abstract settings, such as K-theory, algebraic topology and algebraic geometry, we chose to only tackle the case of de Rham cohomology as it is the most digestible and we believe it is the best at providing intuition behind the more abstract approaches.

We assume that the reader has a strong background in differential calculus and linear algebra, and basic knowledge of topology. The symbol $\subset$ is used for (not necessarily proper) subset inclusion and $\mathbb{N}$ denotes the set of positive integers.

2. Smooth Manifolds and Smooth Maps

In this section we introduce the reader to the differentiable spaces we will be discussing throughout the paper. We already know how to do differential calculus on Euclidean space, so to talk about differentiability on general spaces we want to consider spaces that locally "look like" $\mathbb{R}^n$. In this way, we can transfer the notion of differentiability to our space of interest. One can think of these "locally Euclidean spaces" as spaces where, if you "zoom in" close enough into the space, it looks like an n-tuple of real numbers.

2.1. **Manifolds**

To begin, let M be a topological space and $U, V \subset M$ be open. Two homeomorphisms $\phi : U \to \phi(U) \subset \mathbb{R}^n$, $\psi : V \to \psi(V) \subset \mathbb{R}^n$, where $\phi(U)$ and $\phi(V)$ are open in $\mathbb{R}^n$, are said to be smoothly related if

$$\begin{aligned}\phi \circ \psi^{-1} &: \psi(U \cap V) \to \phi(U \cap V),\\ \psi \circ \phi^{-1} &: \phi(U \cap V) \to \psi(U \cap V),\end{aligned}$$

are C^∞. A (smooth) atlas on M is a collection $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}$ where $\{U_\alpha\}$ is an open cover of M and the homeomorphisms $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha) \subset \mathbb{R}^n$ are all mutually smoothly related. A pair $(U, \phi) \in \mathcal{A}$ is called a chart, U is called a coordinate neighbourhood, and ϕ is called a coordinate map or a chart map. For two charts (U, ϕ), $(V, \psi) \in \mathcal{A}$, we call the composition $\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V)$ a transition map. It's easy to check that every smooth atlas on M defines a maximal smooth atlas on M (just add to the atlas all the charts (U, ϕ) such that the homeomorphisms $\phi : U \to \phi(U) \subset \mathbb{R}^n$ are smoothly related to all elements in the atlas). This allows us to give the definition of a smooth manifold:

Definition 2.1. *A smooth manifold M is a secound countable Hausdorff space M with a maximal smooth atlas $\mathcal{A}$. If every coordinate neighbourhood is homeomorphic to an open subset of $\mathbb{R}^n$, where n is fixed, we call n the dimension of M, denoted* $\dim M$. *If M is a manifold with* $\dim M = n$, *we say that M is a smooth n-manifold. We write M^n if we wish to make the dimension clear.*

Since a smooth atlas defines a maximal smooth atlas, when constructing a manifold one only needs to specify the former.

If we instead consider transition functions that are of class C^k, piece-wise linear, holomorphic, etc, one obtains C^k manifolds, piece-wise linear manifolds, complex manifolds, etc (C^0 manifolds are simply called topological manifolds). We will focus our attention on smooth manifolds and will simply refer to these as "manifolds". We sometimes write coordinate maps $\phi : U \to \phi(U)$ by $(x^1, \ldots, x^n)$, where $x^i = \pi^i \circ \phi : U \to \mathbb{R}$ and $\pi^i : \mathbb{R}^n \to \mathbb{R}$ is the projection onto the ith factor. Similarly, we may also write $(U, x^1, \ldots, x^n)$ for the chart (U, ϕ) with ϕ having coordinate functions $x^1, \ldots, x^n$.

Example 2.2. *One can see that $\mathbb{R}^n$ with the maximal atlas which contains $\{(\mathbb{R}^n, \mathrm{id})\}$, where $\mathrm{id} : \mathbb{R}^n \to \mathbb{R}^n$ is the identity map, is a smooth manifold. Whenever we speak of $\mathbb{R}^n$ as a smooth manifold, it will always be with this structure (called the "usual structure").*

A standard (non-trivial) example of a manifold that may come to mind when reading the definition is the sphere. In our local field of vision here on Earth, assuming we can only walk on the surface of the Earth, the ground resembles a plane (in the sense that we can only walk forwards, backwards, left, right and combinations of these). Of course, it is well-known that the surface of the Earth resembles more to a sphere rather than a plane when seen from space (despite what a non-negligible amount of people may say on the internet). We now generalize this example and explore a few others that will be of interest to us.

Example 2.3 (n-Sphere)**.** *Let $S^n = \{x \in \mathbb{R}^{n+1} : (x^1)^2 + \cdots + (x^{n+1})^2 = 1\}$ be the n-sphere. We let $N = (0, \ldots, 0, 1)$, $S = (0, \ldots, 0, -1)$ be the north and south poles, respectively. We claim that his is an n-dimensional smooth manifold. As a subspace of $\mathbb{R}^n$, this is Hausdorff and second countable. It suffices to define an atlas on S^n. Consider the open sets $U_N = S^n \setminus \{N\}$, $U_S = S^n \setminus \{S\}$. We define the maps $\phi_N : U_N \to \mathbb{R}^n$ and $\phi_S : U_S \to \mathbb{R}^n$ by*

$$\phi_N(x) = \frac{1}{1 - x^{n+1}}(x^1, \ldots, x^n), \quad \phi_S(x) = \frac{1}{1 + x^{n+1}}(x^1, \ldots, x^n),$$

where $x = (x^1, \ldots, x^{n+1})$. One can easily verify that the inverses of the maps are

$$\phi_N^{-1}(x) = \frac{1}{1 + |x|^2}(2x^1, \ldots, 2x^n, |x|^2 - 1), \quad \phi_S^{-1}(x) = \frac{1}{1 + |x|^2}(2x^1, \ldots 2x^n, 1 - |x|^2).$$

From this, we see that ϕ_N and ϕ_S are homeomorphisms. Moreover, the transition map $\phi_N \circ \phi_S^{-1} : \phi_S(U_N \cap U_S) \to \phi_N(U_N \cap U_S)$ is such that

$$(\phi_N \circ \phi_S^{-1})(x) = \frac{x}{|x|^2}$$

(note that, since $\phi_S^{-1}(0) = N \notin U_N$, the denominator is never zero). Hence, $\phi_N \circ \phi_S^{-1}$ is C^∞ and so $\phi_S \circ \phi_N^{-1} = (\phi_N \circ \phi_S^{-1})^{-1}$ is also C^∞.

This shows that S^n is a smooth manifold with the maximal atlas containing $\{(U_N, \phi_N), (U_S, \phi_S)\}$. This maximal atlas is the usual structure on S^n.

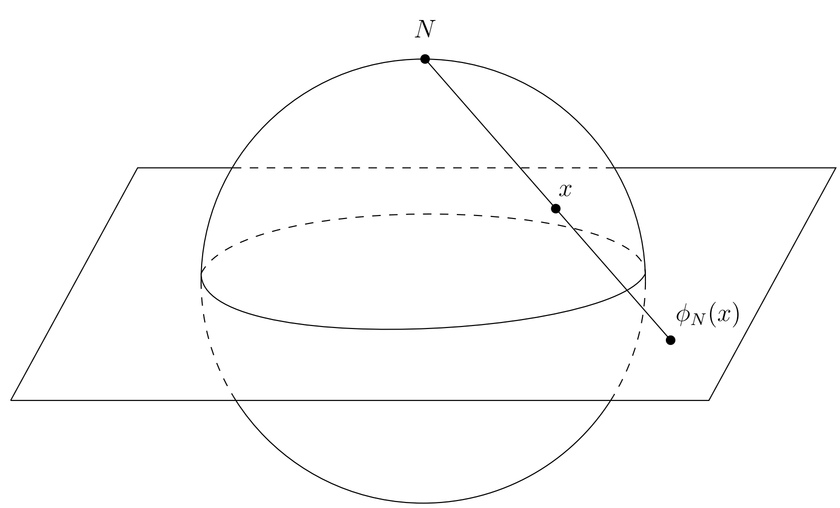

FIGURE 2.1. Stereographic projection of S^2 onto $\mathbb{R}^2$

Example 2.4 (Möbius Strip). *Consider the space $M = [0,1]\times(-1,1)/\sim$, where $\sim$ is generated by $(0,y)\sim(1,-y)$ for all $y\in(-1,1)$. We denote by $[(x,y)]$ the equivalence class with representative (x,y) and $p:[0,1]\times(-1,1)\to M$ the canonical projection. If an open set, U, in M contains the point $[(0,y)]\in M$, then $p^{-1}(U)$ contains $(0,y)$ and $(1,-y)$. Hence, the open sets in M are exactly the sets $p(U)$ where U is open and contains $(1,-y)$ whenever $(0,y)\in p(U)$. With this remark, the Hausdorff and second countable properties easily follow. We now define an atlas on this space.*

Let $V_1=(0,1)\times(-1,1)$, and $V_2=[0,1/2)\times(-1,1)\cup(1/2,1]\times(-1,1)$. Then p is injective on V_1, and $U_1=p(V_1)$ and $U_2=p(V_2)$ are open sets which cover M. Let $q:V_2\to(0,1)\times(-1,1)$ be given by $q(x,y)=(-x+1/2,y)$ if $(x,y)\in[0,1/2)\times(-1,1)$ and $q(x,y)=(-x+3/2,-y)$ if $(x,y)\in(1/2,1]\times(-1,1)$. Then $q(0,y)=q(1,-y)$ and $(x_1,y_1)\sim(x_2,y_2)$ if and only if $q(x_1,y_1)=q(x_2,y_2)$, so this defines a (unique) continuous map $\tilde q: M\to(0,1)\times(-1,1)$ such that $\tilde q\circ p=q$. The map $\tilde q$ is a homeomorphism. We define $\phi_1:U_1\to(0,1)\times(-1,1)$ and $\phi_2:U_2\to(0,1)\times(-1,1)$ by $\phi_1=(p|_{V_1})^{-1}$ and $\phi_2=\tilde q$. Note that $\phi_1^{-1}([(x,y)])=(x,y)$ and so on $\phi_1(U_1\cap U_2)=(0,1/2)\times(-1,1)\cup(1,2,1)\times(-1,1)$,

$$\phi_2\circ\phi_1^{-1}(x,y)=\phi_2([x,y])=\begin{cases}(-x+1/2,y), & (x,y)\in(0,1/2)\times(-1,1)\\(-x+3/2,-y), & (x,y)\in(1/2,1)\times(-1,1)\end{cases}$$

which is smooth. Hence, $\{(U_1,\phi_1),(U_2,\phi_2)\}$ is an atlas of M. The resulting manifold is called the Möbius strip.

Example 2.5 (Product Manifold). *Let $M_1^{n_1}$ and $M_2^{n_2}$ be smooth manifolds with maximal atlases $\{(U_\alpha,\phi_\alpha)\}_{\alpha\in A}$ and $\{(V_\beta,\psi_\beta)\}_{\beta\in B}$ respectively. Then $M_1\times M_2$ is a smooth manifold of dimension n_1+n_2 with the smooth structure induced by the atlas $\{(U_\alpha\times V_\beta,\phi_\alpha\times\psi_\beta)\}_{(\alpha,\beta)\in A\times B}$, where*

$$\phi_\alpha\times\psi_\beta(x,y)=(\phi_\alpha(x),\psi_\beta(y)).$$

Note that the atlas $\{(U_\alpha\times V_\beta,\phi_\alpha\times\psi_\beta)\}_{(\alpha,\beta)\in A\times B}$ is not necessarily maximal (similar to the fact that not every open subset of $\mathbb{R}^2$ is a product of open subsets of $\mathbb{R}$). Using this construction inductively, the finite product of smooth manifold is a smooth manifold as well, whose dimension is the sum of the individual dimensions. A particular example of this is the n-fold product of circles, called the n-torus: $T^n=S^1\times\cdots\times S^1$.

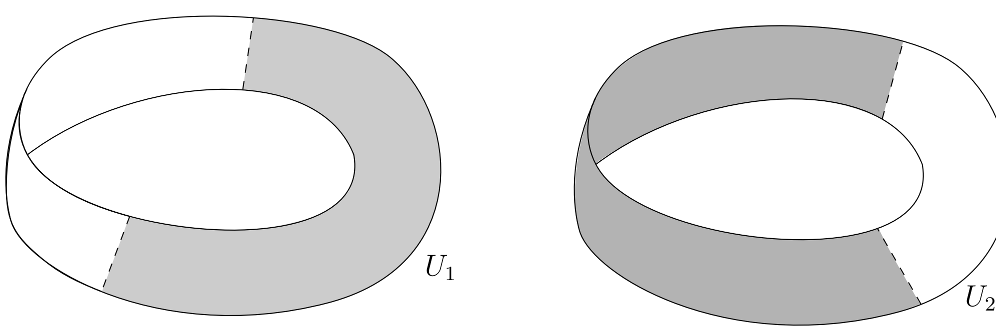

Figure 2.2. Coordinate neighbourhoods of the Möbius strip.

Example 2.6 (Open Submanifolds). *If U is an open subset of a smooth manifold M with a maximal atlas $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in A}$, then U with the atlas $\{(U_\alpha \cap U, \phi_\alpha|_{U_\alpha \cap U}) : U \cap U_\alpha \neq \emptyset\}_{\alpha \in A}$ is a smooth manifold.*

Example 2.7 (Finite Dimensional Vector Spaces). *Let V be an n-dimensional vector space with a norm (and hence a topology). Fix a basis $E_1, \dots, E_n$ for V. This defines an isomorphism $E : V \to \mathbb{R}^n$, $E(E_i) = e_i$, where $e_1, \dots, e_n$ is the standard basis for $\mathbb{R}^n$. This isomorphism is a homeomorphism between the two topological spaces, so V is Hausdorff and second countable. Moreover, $\{(V, E)\}$ is an atlas on V and so it defines a maximal atlas, making V a smooth manifold. Note that this atlas is independent of the choice of basis: if $\tilde{E}_1, \dots, \tilde{E}_n$ is another basis, then there is a matrix $A = (A_i^j)$ with $\tilde{E}_i = \sum_{j=1}^n A_i^j E_j$. Let $\tilde{E} : V \to \mathbb{R}^n$ be the isomorphism induced by the new basis. We see that*

$$\begin{aligned} E \circ \tilde{E}^{-1}(x^1, \dots, x^n) &= E\Big(\sum_{i=1}^n x^i \tilde{E}_i\Big) \\ &= \sum_{i=1}^n x^i E\Big(\sum_{j=1}^n A_i^j E_j\Big) \\ &= \sum_{i,j=1}^n x^i A_i^j e_j \\ &= A(x^1, \dots, x^n). \end{aligned}$$

So $E \circ \tilde{E}^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth map and hence E and $\tilde{E}$ are smoothly compatible, that is, they both define the same maximal atlas. This shows that every basis of V gives the same smooth manifold structure on V.

The last two examples provides us with a large amount of examples of smooth manifolds. The following one is of particular interest.

Example 2.8 (Space of Matrices). *The space $M(n \times m, \mathbb{R})$ of $n \times m$ matrices with entries in $\mathbb{R}$ is a vector space of dimension nm, and hence a smooth manifold of the same dimension. If we restrict our attention to $n \times n$ matrices, then using continuity of the determinant, we find that the open subset $\mathrm{GL}(n, \mathbb{R}) = \det^{-1}(\mathbb{R} \setminus \{0\})$ of invertible $n \times n$ matrices is a smooth manifold.*

Example 2.9 (Projective Space). *Let $\mathbb{R}P^n$ be the set of all 1-dimensional subspaces of $\mathbb{R}^{n+1}$, that is, $\mathbb{R}P^n = (\mathbb{R}^{n+1} \setminus \{0\})/\sim$, where $x \sim y$ if and only if $x = \lambda y$ for some $\lambda \neq 0$. We claim that this is a smooth manifold of dimension n. We can see this space as S^n with the antipodes identified, that is, $\mathbb{R}P^n \cong S^n/\sim$, where $x \sim y$ if and only if $x = -y$. Let $p : S^n \to \mathbb{R}P^n$ be the canonical projection. The open sets in $\mathbb{R}P^n$ are then sets of the form $p(U)$ where U is open and contains $-x$ whenever it contains x. The Hausdorff and second countable properties follow. We put on this space a smooth structure as follows.*

Let $V_1, \dots, V_{n+1}$ be the sets $V_i = \{(x^1, \dots, x^{n+1}) \in S^n : x^i \neq 0\}$. Note that $p(V_i)$ is an open set in $\mathbb{R}P^n$. Define $\phi_i : U_i \to B_1(0)$, where $B_1(0)$ is the open ball of 1 *centered at 0 in* $\mathbb{R}^n$*, by*

$$\phi_i([x^1,\ldots,x^{n+1}]) = \frac{|x^i|}{x^i}(x^1,\ldots,\hat{x}^i,\ldots,x^{n+1}) = \phi_i([-x^1,\ldots,-x^{n+1}]),$$

for $i = 1,\ldots,n+1$ *(the hat symbol denotes the omission of* x^i*). It is easy to see that this map is a homeomorphism with inverse*

$$\phi_i^{-1}(x^1,\ldots,x^n) = [x^1,\ldots,x^{i-1},\sqrt{1-|x|^2},x^i,\ldots,x^n].$$

where $x = (x^1,\ldots,x^n)$*. Indeed,*

$$\begin{aligned}
(\phi_i\circ\phi_i^{-1})(x^1,\ldots,x^n) &= \phi_i([x^1,\ldots,x^{i-1},\sqrt{1-|x|^2},x^i,\ldots,x^n]) \\
&= \frac{\sqrt{1-|x|^2}}{\sqrt{1-|x|^2}}(x^1,\ldots,x^{i-1},x^i,\ldots,x^n) \\
&= (x^1,\ldots,x^n), \\
(\phi_i^{-1}\circ\phi_i)([x^1,\ldots,x^{n+1}]) &= \phi_i^{-1}\Big(\frac{|x^i|}{x^i}(x^1,\ldots,\hat{x}^i,\ldots,x^{n+1})\Big) \\
&= \frac{|x^i|}{x^i}[x^1,\ldots,x^{i-1},\frac{x^i}{|x^i|}\sqrt{1-|x|^2+(x^i)^2},x^{i+1},\ldots,x^{n+1}] \\
&= \frac{|x^i|}{x^i}[x^1,\ldots,x^{i-1},x^i,x^{i+1},\ldots,x^{n+1}] \\
&= [x^1,\ldots,x^{n+1}].
\end{aligned}$$

The last line occurs because $|x^i|/x^i = \pm 1$*, but in either case, the two points correspond to the same equivalence class in* $\mathbb{R}P^n$*. The transition maps are given by*

$$(\phi_i\circ\phi_j^{-1})(x^1,\ldots,x^n) = \frac{|x^i|}{x^i}(x^1,\ldots,x^{i-1},x^{i+1},\ldots,x^{j-1},\sqrt{1-|x|^2},x^j,\ldots,x^n)$$

where $x = (x^1,\ldots,x^n) \in U_i\cap U_j$*,* $i<j$*. This is smooth since* $x^i, x^j \neq 0$ *on* $U_i\cap U_j$*. This shows that* $\mathbb{R}P^n$ *is a smooth manifold. If instead we would consider all the 1-dimensional complex subspaces of* $\mathbb{C}^{n+1}$*, we would obtain in the complex projective space,* $\mathbb{C}P^n$*, with dimension* $2n$ *(as a real manifold).*

This example is of crucial importance to us: it is the first example of a Grassmann manifold (or more simply, a Grassmannian). These manifolds allows us to construct a manifold, called the universal bundle, which is so twisted that any vector bundle is a pullback of this universal bundle (pullbacks of vector bundles usually give bundles that are less twisted, as we will see later).

2.2. **Smooth Maps**

In order to study topological spaces, one has the notion of continuous functions (i.e., functions that "detect" the topology of the space), similarly to study groups or other algebraic structures, one has different maps that behave in a "compatible" way with the algebraic structure. To study smooth manifolds, we want to consider maps that detect both the topology (since manifolds are topological spaces) and the smooth structure. Moreover, we would also like this more general notion of differentiability to reduce to that of $\mathbb{R}^n$ which we already know. Thankfully, charts will make this job possible.

Definition 2.10. *Let M be a smooth manifold. A real-valued function $f : M \to \mathbb{R}$ is said to be a smooth function on M if there exists an atlas $\mathcal{A}$ for M such that, for all charts $(U, \phi) \in \mathcal{A}$, $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}$ is a smooth function on $\phi(U) \subset \mathbb{R}^n$. We write $C^\infty(M)$ for the set of all smooth functions on M.*

Note that, if $f : M \to \mathbb{R}$ is a smooth function on M, then $f \circ \psi^{-1} : \psi(V) \to \mathbb{R}$ is smooth for any chart (V, ψ) in the maximal atlas for M. By Example 2.6, any open subset of M is also a smooth manifold, so we can speak of smooth functions on open subsets of M. For example, if $(U, x^1, \dots, x^n)$ is a chart on M, each $x^i : U \to \mathbb{R}$ is a smooth function on U.

We can see that this notion of differentiability reduces to that of multivariable calculus: consider the atlas $\{(\mathbb{R}^n, \mathrm{id})\}$ on $\mathbb{R}^n$, if f is smooth in the sense of the definition above, then $f \circ \mathrm{id}^{-1} = f \circ \mathrm{id} = f$ is a smooth function on $\mathrm{id}(\mathbb{R}^n) = \mathbb{R}^n$ in the sense of multivariable calculus.

If $f, g \in C^\infty(M)$ and $a \in \mathbb{R}$, then $f + g$ given by $(f + g)(p) = f(p) + g(p)$, fg given by $(fg)(p) = f(p)g(p)$ and af given by $(af)(p) = af(p)$ are all C^∞ functions on M. This turns $C^\infty(M)$ into an algebra over $\mathbb{R}$ (recall that an algebra over a field $\mathbb{F}$ is a vector space V over $\mathbb{F}$ with a multiplication map that turns V into a ring with $a(v_1 \cdot v_2) = (av_1) \cdot v_2 = v_1 \cdot (av_2)$, for $v_1, v_2 \in V$, $a \in \mathbb{F}$).

Definition 2.11. *We say that a continuous map $f : M \to N$ is a smooth map if there exists atlases $\mathcal{A}_M$ and $\mathcal{A}_N$ for M and N, respectively, such that for any choice of charts (U, ϕ) in $\mathcal{A}_M$ and (V, ψ) in $\mathcal{A}_N$, the composition*

$$\psi \circ f \circ \phi^{-1} : \phi(U \cap f^{-1}(V)) \to \psi(V)$$

is smooth as a map between Euclidean spaces.

Note that, since f is continuous, $f^{-1}(V)$ is open in M and $\phi(U \cap f^{-1}(V))$ is open as a subset of the corresponding Euclidean space, so it's alright to talk about the smoothness of $\psi \circ f \circ \phi^{-1}$ in the usual sense of multivariable calculus. We will use "differentiable" and "C^∞" as synonyms of "smooth".

Of course, this notion of differentiability for a map $f : M \to N$ between manifolds coincides with the well-established definition from multivariable calculus when $M = \mathbb{R}^n$ and $N = \mathbb{R}^m$. One can easily show that the composition of smooth maps is also smooth and that the restriction of a smooth map to an open subset is also smooth. The identity map is smooth since coordinate maps are smoothly related.

Definition 2.12. *A bijective map $f : M \to N$ between manifolds is a diffeomorphism if both f and f^{-1} are smooth.*

Example 2.13. *Let (U, ϕ) be a chart on a smooth manifold M. Then $\phi : U \to \phi(U) \subset \mathbb{R}^n$ is a diffeomorphism of U onto $\phi(U)$.*

Theorem 2.14. *Let M and N be smooth manifolds and let $\{U_\alpha\}$ be an open cover of M. Suppose that, for each α, we have smooth maps $f_\alpha : U_\alpha \to N$ such that $f_\alpha|_{U_\alpha \cap U_\beta} = f_\beta|_{U_\alpha \cap U_\beta}$ for each α and β. Then there is a global smooth map $f : M \to N$ such that $f|_{U_\alpha} = f_\alpha$.*

Proof. Let $f : M \to N$ be given by $f(p) = f_\alpha(p)$ for $p \in U_\alpha$. By the gluing lemma, f is continuous. Moreover, if $p \in M$, then $p \in U_\alpha$ for some α, so there is a chart (U, ϕ) for M with $p \in U \subset U_\alpha$ and a chart (V, ψ) for N with $f(p) \in V$. It follows that $\psi \circ f \circ \phi = \psi \circ f_\alpha \circ \phi^{-1}$ on $\phi(U \cap f^{-1}(V)) = \phi(U \cap U_\alpha \cap f^{-1}(V))$. Since f_α is smooth, we conclude that f is also smooth. □

We know $\mathbb{R}^n$ quite well and know of many different constructions on this space, so we would of course like to find a way to extend these constructions to manifolds by using their local structure. In order to do this, we introduce the notion of partitions of unity, which will be useful in our study of manifolds.

Recall that the support of a function $f : M \to \mathbb{R}$ is the set

$$\mathrm{cl}_M(\{x \in M : f(x) \neq 0\}).$$

Definition 2.15. *A partition of unity on a manifold M is a collection of smooth functions $\{\psi_\alpha\}_{\alpha \in A}$, $\psi_\alpha : M \to \mathbb{R}$, such that*

1. $0 \leq \psi_\alpha(p) \leq 1$ *for all* $\alpha \in A$ *and* $p \in M$,
2. *for each point* $p \in M$, $\{\operatorname{supp} \psi_\alpha\}$ *is locally finite, that is, every* $p \in M$ *has a neighbourhood that intersects only finitely many of the* $\operatorname{supp} \psi_\alpha$, *and*
3. *for each* $p \in M$, $\sum_\alpha \psi_\alpha(p) = 1$.

Proposition 2.16 (Existence of Partitions of Unity)**.** *Given an open cover $\{U_\alpha\}_{\alpha \in A}$ of a smooth manifold M, there is a countable partition of unity $\{\psi_i\}_{i \in \mathbb{N}}$ such that the support of ψ_i is compact and is contained in U_α for some $\alpha \in A$. If we do not assume compact support, then there is a partition of unity $\{\psi_\alpha\}$ with $\operatorname{supp} \psi_\alpha \subset U_\alpha$ for each $\alpha \in A$ with at most countably many of the ψ_α not identically zero (in this case we say that $\{\psi_\alpha\}$ is subordinate to $\{U_\alpha\}$).*

Proof. See Theorem 1.11 in [6]. □

This result leads us to two very useful corollaries.

Corollary 2.17. *Given a closed subset A of a smooth manifold M, and any open set U containing A, there exists a smooth function $b : M \to \mathbb{R}$ such that $0 \leq b(x) \leq 1$ for all $x \in M$, $b(x) = 1$ for all $x \in A$, and $\operatorname{supp} b \subset U$.*

Proof. Let $\{\psi_1, \psi_2\}$ be a partition of unity subordinate to $U_1 = U$ and $U_2 = M \setminus A$. Then $b = \psi_1$ is the desired function. □

Corollary 2.18. *Let M be a smooth manifold, U an open neighbourhood of some $p \in M$ and $f : U \to \mathbb{R}$ a smooth function. Then there is an open neighbourhood V of p with $\mathrm{cl}_M(V) \subset U$ and a smooth function $\tilde{f} : M \to \mathbb{R}$ such that $\tilde{f}|_V = f|_V$ and $\tilde{f}(q) = 0$ if $q \notin U$.*

Proof. Let (W, ϕ) be a chart with $W \subset U$, $\phi(p) = 0$ and $B_2(0) \subset \phi(W)$ (this always exists by composing with translations and scalar multiplications). Then $V = \phi^{-1}(B_1(0))$ is such that $\mathrm{cl}(V) \subset W \subset U$. By the previous corollary, there exists a smooth function $b : M \to \mathbb{R}$ such that $b(x) = 1$ for all $x \in \mathrm{cl}_M(V)$ and $\operatorname{supp} b \subset W$. Define $\tilde{f} : M \to \mathbb{R}$ by

$$\tilde{f}(x) = \begin{cases} b(x) f(x), & x \in W, \\ 0, & x \notin W. \end{cases}$$

This is the desired function. □

This, in particular, means we can extend a coordinate map $\phi : U \to \phi(U)$ to a smooth map $\tilde{\phi} : M \to \mathbb{R}^n$, by shrinking U beforehand and extending each of the coordinate functions $x^i : U \to \mathbb{R}$ to smooth functions on M. Of course, this is no longer a diffeomorphism.

2.3. **Submanifolds**

We conclude this section by briefly discussing submanifolds of a smooth manifold M.

Definition 2.19. *A subset $N \subset M$ is a k-dimensional submanifold of M if for any $p \in N$, there is a chart $(U, x^1, \ldots, x^n)$ of M with $p \in U$ such that*

$$N \cap U = \{q \in U : x^{k+1}(p) = \cdots = x^n(p) = 0\}.$$

Moreover, if N is a closed subset of M, we call N a closed submanifold.

Note that a k-dimensional submanifold N of M has a smooth structure making N a k-dimensional smooth manifold. If $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in A}$ is an atlas for M, and $B \subset A$ is the set of all $\beta \in A$ such that $(U_\beta, x^1_\beta, \ldots, x^n_\beta)$ satisfies

$$N \cap U_\beta = \{q \in U_\beta : x^{k+1}_\beta(q) = \cdots = x^n_\beta(q) = 0\},$$

then $\{(N \cap U_\beta, x^1_\beta, \ldots, x^k_\beta)\}_{\beta \in B}$ is an atlas for N.

If the reader has had previous experience with manifolds as subsets of $\mathbb{R}^n$, then the above definition might seem familiar. One can see that open subsets of manifolds are submanifolds of the same dimension, and the sphere S^n is a closed submanifold of $\mathbb{R}^{n+1}$.

Example 2.20. *Consider the projective n-space $\mathbb{R}P^n$. This can be seen as a quotient space of $\mathbb{R}^{n+1}$ (see Example 2.9). The inclusion $i : \mathbb{R}^{n+1} \hookrightarrow \mathbb{R}^{n+2}$ induces a smooth map $\iota : \mathbb{R}P^n \hookrightarrow \mathbb{R}P^{n+1}$ such that the diagram*

$$\begin{tikzcd}
\mathbb{R}^{n+1} \arrow[r, hook, "i"] \arrow[d, "p"'] & \mathbb{R}^{n+2} \arrow[d, "p"] \\
\mathbb{R}P^n \arrow[r, hook, "\iota"] & \mathbb{R}P^{n+1}
\end{tikzcd}$$

commutes. So we can consider $\mathbb{R}P^n$ as a submanifold of $\mathbb{R}P^{n+1}$. We can then define $\mathbb{R}P^\infty$ as the union of all the $\mathbb{R}P^n$, and define a set $U \subset \mathbb{R}P^\infty$ to be open if $U \cap \mathbb{R}P^n$ is open for all n (i.e. we put on $\mathbb{R}P^\infty$ the weak topology). This procedure can also be done if we replace $\mathbb{R}$ by $\mathbb{C}$.

3. Bundles Over Manifolds

3.1. **Vector Bundles**

The usual way one pictures a vector $v = (v^1, \dots, v^n)$ in $\mathbb{R}^n$ is as an arrow starting from the origin and terminating at the point $v \in \mathbb{R}^n$, but one can very well picture v as starting at some other point $p \in \mathbb{R}^n$ and terminating at the point $(p^1+v^1, \dots, p^n+v^n)$. Let us write v_p for the vector v pictured as an arrow starting at p. If one considers the set of all such v_p, then one can naturally put a vector space structure on this set: if $u, v \in \mathbb{R}^n$, $a \in \mathbb{R}$, one can define addition by $u_p + v_p = (u+v)_p$ and scalar multiplication by $av_p = (av)_p$. Let us make this idea more precise and in a way that can be generalized to manifolds.

For $p \in \mathbb{R}^n$, let $\mathbb{R}^m_p = \{p\} \times \mathbb{R}^m$ (we view this as the set of all m-dimensional arrows starting at p). Taking the union over all such $p \in \mathbb{R}^n$, we have

$$E = \bigcup_{p \in \mathbb{R}^n} \mathbb{R}^m_p = \mathbb{R}^n \times \mathbb{R}^m$$

(note that this is in fact a disjoint union). We naturally have a projection map $\pi : E \to \mathbb{R}^n$, given by $\pi(p, v) = p$, which is smooth. We can further endow $\pi^{-1}(p) = \mathbb{R}^m_p$ with the structure of a real vector space: define $\oplus : \bigcup_{p \in \mathbb{R}^n} \mathbb{R}^m_p \times \mathbb{R}^m_p \to E$ and $\odot : \mathbb{R} \times E \to E$ by

$$(p, v) \oplus (p, w) = (p, v + w),$$
$$c \odot (p, v)(p, cv).$$

We will usually write $\oplus$ and $\odot$ as $+$ and $\cdot$ (or just by concatenation).

One particular case of interest is when $m = n$. We call $\mathbb{R}^n_p$ the tangent space at p, usually denoted by $T_p\mathbb{R}^n$, and $E = \bigcup_{p \in \mathbb{R}^n} \mathbb{R}^n_p$ the tangent bundle of $\mathbb{R}^n$, denoted $T\mathbb{R}^n$. A smooth map $f : \mathbb{R}^n \to \mathbb{R}^m$ induces a linear map $f_{*,p} : T_p\mathbb{R}^n \to T_{f(p)}\mathbb{R}^m$ by $f_{*,p}(v_p) = (Df(p)(v))_{f(p)}$, where $Df(p)$ the derivative of f at p. The reason this is of interest is because differential calculus allows us to gain insight of maps $f : \mathbb{R}^n \to \mathbb{R}^m$ using its derivative, $Df(p) : \mathbb{R}^n \to \mathbb{R}^m$. This insight comes from the various theorems one learns in undergraduate mathematics: the mean value theorem, inverse and implicit function theorems, constant rank theorems, etc. More than that, we can even gain geometric information about the domain of f using this map of tangent spaces. For example, it is quite easy to show that $\mathbb{R}^n$ and $\mathbb{R}^m$ can only be diffeomorphic if $n = m$ by reducing this problem to a linear algebra one using the derivative, a result which is much far trivial if we replace "diffeomorphic" by "homeomorphic". This highlights the perks of differential calculus: we can reduce problems about the topology of our spaces to linear algebra problems. If $U \subset \mathbb{R}^n$ is open, we can define

$$T_pU = \{p\} \times \mathbb{R}^n, \quad \text{and} \quad TU = \bigcup_{p \in U} \mathbb{R}^n_p$$

and the discussion above carries over.

We can take advantage of the fact that manifolds have open subsets which are diffeomorphic to open subsets of $\mathbb{R}^n$ in order to generalize the above.

Definition 3.1. *A (smooth) real vector bundle of rank k over a manifold B is a quintuple $(E, B, \pi, \oplus, \odot)$ with*

1. *E a manifold and $\pi : E \to B$ a smooth onto map,*
2. *the maps*
$$\oplus : \bigcup_{p \in B} \pi^{-1}(p) \times \pi^{-1}(p) \to E \quad \text{such that} \quad \oplus\left(\pi^{-1}(p) \times \pi^{-1}(p)\right) \subset \pi^{-1}(p),$$
$$\odot : \mathbb{R} \times E \to E \qquad \text{such that} \quad \odot\left(\mathbb{R} \times \pi^{-1}(p)\right) \subset \pi^{-1}(p)$$
turn $\pi^{-1}(p)$, called the fibre over p, into a real vector space of dimension k,
3. *the following local triviality holds: for any $p \in B$, there exists an open neighbourhood U of p and a diffeomorphism $\psi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ such that*
$$\psi|_{\pi^{-1}(q)} : \pi^{-1}(q) \to \{q\} \times \mathbb{R}^k$$
is a linear isomorphism, for each $q \in U$. We call ψ a trivialization for E and U a trivializing open set for E.

The spaces E and B are called the total space and the base space respectively, and π the projection map.

If $v, w \in \pi^{-1}(p)$, we usually write $v \oplus w = v + w$ and similarly for $\odot$. We usually refer to a vector bundle $(E, B, \pi, +, \cdot)$ simply as $\pi : E \to B$. We also write $E_p = \pi^{-1}(p)$. If we replace $\mathbb{R}$ by $\mathbb{C}$ we obtain a complex vector bundle of rank k. A vector bundle of rank 1 is called a line bundle.

Example 3.2. *As we previously discussed, $E = \mathbb{R}^n \times \mathbb{R}^k$, $B = \mathbb{R}^n$ and the projection $\pi : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ form a vector bundle of rank k.*

Example 3.3. *If M is an n-dimensional manifold, $E = M \times \mathbb{R}^k$ is a vector bundle of rank k over M with projection map $\pi(p, v) = p$ for $p \in M$, $v \in \mathbb{R}^k$. We call this the product bundle of rank k over M.*

Example 3.4. *If $\pi : E \to B$ is a vector bundle and $U \subset B$ an open subset, then U inherits a vector bundle structure with total space $\pi^{-1}(U)$, projection $\pi|_{\pi^{-1}(U)}$, and addition and scalar multiplication using the appropriate restrictions.*

Before introducing our next example of a vector bundle, let us take a more abstract look at $T_p\mathbb{R}^n$, $p \in \mathbb{R}^n$. Let $v \in \mathbb{R}^n$, $v = \sum_{i=1}^n v^i e_i$, where $e_1, \ldots, e_n$ is the standard basis of $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$ a smooth function. Then $v_p = \sum_{i=1}^n v^i (e_i)_p$. We have a map $T_p\mathbb{R}^n \to \mathbb{R}$ given by
$$v_p \longmapsto D_v f(p) = \sum_{i=1}^n v^i D_i f(p),$$
i.e. the directional derivative of f in the direction of v at p. If $f, g \in C^\infty(\mathbb{R}^n)$, $D_v(f + g) = D_v f + D_v g$, also by the product rule,
$$D_v(fg) = f(p) D_v g + g(p) D_v f,$$
and if $a \in \mathbb{R}$, then $D_v(af) = aD_v(f)$. A map $C^\infty(\mathbb{R}^n) \to \mathbb{R}$ satisfying these conditions is called a linear derivation at p. These two ways of looking at v, either as a pair (p, v) or as a linear derivation at p, are in fact equivalent (see Section 2 of [7] for a more complete treatment of this). The advantage of this latter perspective is that

it allows us to have an intrinsic definition of tangent vectors at a point without any reference to a basis of $T_p\mathbb{R}^n$, and this is much easier to transfer to general manifolds.

Definition 3.5. *Let M be a manifold and $p \in M$. The tangent space at p, denoted T_pM, is the set of all maps $v : C^\infty(M) \to \mathbb{R}$ satisfying*

(1) $v(f+g) = v(f) + v(g)$,
(2) $v(af) = av(f)$,
(3) $v(fg) = f(p)v(g) + g(p)v(f)$,

where $f, g \in C^\infty(M)$ and $a \in \mathbb{R}$.

We sometimes write $v \in T_pM$ as v_p to make the base point explicit. Note that T_pM has a natural vector space structure given by point-wise addition and scalar multiplication. With this structure, we call T_pM the tangent space of M at p and call $v \in T_pM$ a tangent vector at p.

Let (U, ϕ) be a chart on M and $x^1, \dots, x^n$ be the coordinate functions of ϕ. Then the map

$$C^\infty(M) \ni f \longmapsto D_i(f \circ \phi^{-1})(\phi(p))$$

defines a tangent vector at p, denoted $\partial/\partial x^i|_p$, that is,

$$\left(\frac{\partial}{\partial x^i}\right)\bigg|_p(f) = D_i(f \circ \phi^{-1})(\phi(p)).$$

We usually write $(\partial/\partial x^i)|_p(f) = \partial f/\partial x^i(p)$ or also $(\partial/\partial x^i)|_p(f) = (\partial f/\partial x^i)|_p$. From the case $M = \mathbb{R}^n$, the following fact is easy to accept:

Theorem 3.6. *The tangent space T_pM at $p \in M$ is an n-dimensional vector space. Moreover, if $(U, x^1, \dots, x^n)$ is a chart around p, then $\partial/\partial x^1, \dots, \partial/\partial x^n$ is a basis of T_pM.*

Proof. See Theorem 1.33 in [2]. □

Theorem 3.7. *Let (U, ϕ) and (V, ψ) be two charts around $p \in M$ and let $x^1, \dots, x^n$ and $y^1, \dots, y^n$ be their respective coordinate functions. Then*

$$\frac{\partial}{\partial x^i}\bigg|_p = \sum_{j=1}^n \frac{\partial y^j}{\partial x^i}(p)\frac{\partial}{\partial y^j}\bigg|_p.$$

Proof. This is just an application of the chain rule:

$$\begin{aligned}
\frac{\partial f}{\partial x^i}(p) &= D_i(f \circ \phi^{-1})(\phi(p)) \\
&= D_i(f \circ \psi^{-1} \circ \psi \circ \phi^{-1})(\phi(p)) \\
&= \sum_{j=1}^n D_j(f \circ \psi^{-1})(\psi(p)) D_i(\psi \circ \phi^{-1})^j(\phi(p)) \\
&= \sum_{j=1}^n \frac{\partial f}{\partial y^j}(p)\frac{\partial y^j}{\partial x^i}(p) \\
&= \left(\sum_{j=1}^n \frac{\partial y^j}{\partial x^i}(p)\frac{\partial}{\partial y^j}\bigg|_p\right) f
\end{aligned}$$

□

Let $X_p \in T_pM$. A smooth map $f : M \to N$ induces a linear map $f_{*,p} : T_pM \to T_{f(p)}N$ given by

$$(f_{*,p}X_p)g = X_p(g \circ f),$$

for $g \in C^\infty(N)$. This is indeed a derivation at $f(p)$: let $g, h \in C^\infty(N)$, then

$$\begin{aligned}(f_{*,p}X_p)(gh) &= X_p((gh) \circ f) = X_p((g \circ f) \cdot (h \circ f)) \\ &= g(f(p))X_p(h \circ f) + h(f(p))X_p(g \circ f) \\ &= g(f(p))(f_{*,p}X_p)h + h(f(p))(f_{*,p}X_p)g.\end{aligned}$$

The other properties are proved similarly. We sometimes write f_* for the union of all the $f_{*,p}$.

Definition 3.8. *We call $f_{*,p}$ the pushforward, or differential, of f at p, and f_* the pushforward of f.*

Theorem 3.9. *Let $f : M \to N$ and $g : N \to P$ be smooth maps, and $X_p \in T_pM$. Then*

$$(g \circ f)_{*,p}X_p = g_{*,f(p)}(f_{*,p}X_p).$$

Proof. Let $h \in C^\infty(P)$. Then

$$\begin{aligned}((g \circ f)_{*,p}X_p)h &= X_p(h \circ g \circ f) = f_{*,p}X_p(h \circ g) \\ &= (g_{*,f(p)}(f_{*,p}X_p))h\end{aligned}$$

□

The connection between this more general definition of f_* for manifolds and that for $\mathbb{R}^n$ becomes more evident when seeing the local coordinate description of f_*. We refer the reader to Chapter 3 of [1] for an in-depth look at the tangent space and computations in local coordinates.

We are now ready to construct one of the most important examples of vector bundles. This will generalize the tangent bundle of an open subset of $\mathbb{R}^n$ we previously discussed.

Example 3.10 (Tangent Bundle)**.** *The tangent bundle of a smooth manifold M is the space*

$$TM = \bigcup_{p \in M} T_pM.$$

We have a natural map $\pi : TM \to M$ by $\pi(v_p) = p$ for $v_p \in T_pM$. The topology on TM is given as follows. Let $(U, \phi) = (U, x^1, \ldots, x^n)$ be a chart on M. Since $\partial/\partial x^1|_p, \ldots, \partial/\partial x^n|_p$ is a basis of T_pM for any $p \in U$, if $v_p \in T_pM$, we can write $v_p = \sum_{i=1}^n v^i(\partial/\partial x^i)|_p$. Since this can be done for any $p \in U$, the coefficients $v^1, \ldots, v^n$ depend on v_p, so these are functions on $\pi^{-1}(U)$. This defines a map $\tilde{\phi} : \pi^{-1}(U) \to \phi(U) \times \mathbb{R}^n$ given by

$$\tilde{\phi}(v_p) = (x^1(p), \ldots, x^n(p), v^1(v_p), \ldots, v^n(v_p)) = (x^1 \circ \pi, \ldots, x^n \circ \pi, v^1, \ldots, v^n)(v_p).$$

The inverse of this map is given by

$$\tilde{\phi}^{-1}(\phi(p), c^1, \ldots, c^n) = \sum_{i=1}^n c^i \frac{\partial}{\partial x^i}\bigg|_p.$$

We define a set to be open in $\pi^{-1}(U)$ if and only if $\tilde{\phi}(A)$ is open in $\phi(U) \times \mathbb{R}^n$ (which is an open subset of $\mathbb{R}^{2n}$). The collection of all open sets in $\pi^{-1}(U_\alpha)$, as U_α runs over the coordinate neighbourhoods in the maximal atlas of M, is a basis for the topology of TM. It's easy to see that this topology is Hausdorff and second countable. It is easy to show that $\{(\pi^{-1}(U_\alpha), \tilde{\phi}_\alpha)\}$ is an atlas for TM, making this a smooth manifold.

We still don't have a vector bundle structure on TM, since we are still missing the trivialization maps. The reader might have already guessed what these will be: we let $\psi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ be given by $\psi(v_p) = (\pi(v_p), v^1(v_p), \ldots, v^n(v_p))$, where (U, ϕ) is a chart on M. All the desired properties of a vector bundle are easily checked.

The tangent bundle of a manifold is a natural place to study the geometric properties of the manifold, as it encodes information such as orientation and curvature. Moreover, a smooth map $f : M \to N$ defines a smooth, fibre-preserving, map $f_* : TM \to TN$ such that f_* is linear on each fibre (that is, $f_{*,p} : T_pM \to T_{f(p)}N$ is linear).

Definition 3.11. *Let $\pi_1 : E_1 \to B_1$ and $\pi_2 : E_2 \to B_2$ be vector bundles. A bundle map is a pair $(\tilde{f}, f)$ of smooth maps $\tilde{f} : E_1 \to E_2$, $f : B_1 \to B_2$, such that the following diagram commutes*

$$\begin{tikzcd} E_1 \arrow[r, "\tilde{f}"] \arrow[d, "\pi_1"] & E_2 \arrow[d, "\pi_2"] \\ B_1 \arrow[r, "f"] & B_2 \end{tikzcd}$$

and $\tilde{f}|_{\pi_1^{-1}(p)} : \pi_1^{-1}(p) \to \pi_2^{-1}(f(p))$ is a linear isomorphism for each $p \in B$.

If $B_1 = B_2 = B$, $f = \mathrm{id}_B$ and $\tilde{f} : E_1 \to E_2$ is a diffeomorphism, we call $\tilde{f}$ a bundle equivalence. In this case, we say that $\pi_1 : E_1 \to B$ and $\pi_2 : E_2 \to B$ are equivalent.

Example 3.12. *Any vector bundle $\pi : E \to B$ of rank k equivalent to the product bundle $p : B \times \mathbb{R}^k \to B$ is called a trivial bundle. So the local triviality condition in the definition of a vector bundle says that B has an open cover $\{U_\alpha\}$ such that $\pi^{-1}(U_\alpha)$ is a trivial bundle for each α.*

Definition 3.13. *Let $\pi : E \to B$ be a vector bundle of rank k. A smooth map $s : B \to E$ such that $\pi \circ s = \mathrm{id}_B$ is called a smooth section of E. We write $\Gamma(E)$ for the set of all sections of E.*

Example 3.14. *A smooth section $s : M \to TM$ of the tangent bundle is called a smooth vector field.*

One can replace smooth by continuous in the definition above and obtain what is called a continuous section. Since we will mainly be in the smooth category, we will only restrict to smooth sections unless otherwise stated.

Example 3.15. *Let $\pi : E \to B$ be a vector bundle. There is always a trivial section, $s_0 : B \to E$ given by $s_0(0_p)$, where $0_p \in \pi^{-1}(p)$ is the zero vector. This is also called the zero section of E. Although this section might seem uninteresting, it has an important use.*

Recall that a deformation retraction of a topological space X onto a subspace A of X is a continuous map $F : X \times [0,1] \to X$ such that $F(x,0) = x$, $F(x,1) \in A$ and $F(a,1) = a$ for all $a \in A$. Since we care about smooth manifolds, if X is a smooth manifold, we can consider F to be a smooth map and call this a smooth deformation retraction. We claim that any vector bundle $\pi : E \to B$ deformation retracts onto $s_0(B)$ (and in turn, $s_0(B)$ is diffeomorphic to B). We define $F : E \times [0,1] \to E$ by $F(v_p, t) = (1-t)v_p$. Then $F(v_p, 0) = v_p$, $F(v_p, 1) = 0_p$, and $F(0_p, 1) = 0_p$, so this is the desired deformation retraction.

Defining addition and scalar multiplication of sections point-wise, we can make $\Gamma(E)$ into a vector space (over $\mathbb{R}$ if E is a real vector bundle and over $\mathbb{C}$ if E is a complex vector bundle). Moreover, if $f \in C^\infty(B)$, $s \in \Gamma(E)$, we can define $(fs)(p) = f(p)s(p)$ so that $\Gamma(E)$ becomes a module over $C^\infty(B)$.

Definition 3.16. *A frame field over an open subset $U \subset B$ is a set of k sections $s_1, \ldots, s_k : U \to E$ such that $s_1(p), \ldots, s_k(p)$ forms a basis of E_p, for all $p \in U$.*

Example 3.17. *Let M be a smooth manifold. If $(U, x^1, \ldots, x^n)$ is a chart on M, then $\partial/\partial x^1, \ldots, \partial/\partial x^n : U \to TM$ is a frame field over U.*

Remark 3.18. *If $\pi : E \to M$ is a vector bundle of rank k which has a frame field $s_1, \ldots, s_n$ over the whole of M, then $\pi : E \to M$ is a trivial bundle. In particular, if E is a line bundle, then the existence of a section $s : M \to E$ with $s(p) \neq 0$ for all p (that is, a non-vanishing section) implies that E is trivial.*

We conclude this section by giving an easier way to construct vector bundles. Let $\pi : E \to B$ be a vector bundle, $\{U_\alpha\}_{\alpha \in A}$ an open cover of B where each U_α is a trivializing open set for E, and let $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ trivialization map for each $\alpha \in A$. Then, for $\beta \in A$, $p \in U_\alpha \cap U_\beta$, $v \in \mathbb{R}^n$,

$$\psi_\alpha \circ \psi_\beta^{-1}(p, v) = (p, g_{\alpha\beta}(p)v)$$

where $g_{\alpha\beta} \in \mathrm{GL}(k, \mathbb{R})$. So this defines a smooth map $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(k, \mathbb{R})$, called a transition function. If $\gamma \in A$, we can define new transition functions $g_{\alpha\gamma}$ and $g_{\beta\gamma}$. It is easy to check that these transition functions satisfy the cocycle condition:

$$g_{\alpha\beta}(p)g_{\beta\gamma}(p) = g_{\alpha\gamma}(p)$$

for each $p \in U_\alpha \cap U_\beta \cap U_\gamma$. Moreover, $g_{\alpha\alpha}(p) = \mathrm{id}_{\mathbb{R}^k}$ and $g_{\alpha\beta}(p) = (g_{\beta\alpha}(p))^{-1}$. Conversely, given an open cover $\{U_\alpha\}_{\alpha \in A}$ of B with a family of maps $\{g_{\alpha\beta}\}_{\alpha,\beta \in A}$ with values in $\mathrm{GL}(k, \mathbb{R})$ satisfying

$$g_{\alpha\alpha}(p) = \mathrm{id}_{\mathbb{R}^k}, \quad g_{\alpha\beta}(p) = (g_{\beta\alpha}(p))^{-1}, \quad \text{and} \quad g_{\alpha\beta}(p)g_{\beta\gamma}(p) = g_{\alpha\gamma}(p),$$

for $p \in U_\alpha$, $U_\alpha \cap U_\beta$ and $U_\alpha \cap U_\beta \cap U_\gamma$, respectively, we can construct a vector bundle of rank k by patching the $U_\alpha \times \mathbb{R}^k$ together. We call the family $\{g_{\alpha\beta}\}$ a cocycle.

Definition 3.19. *Two cocycles $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ said to be equivalent if there exist maps $\lambda_\alpha : U_\alpha \to \mathrm{GL}(k, \mathbb{R})$ such that $g_{\alpha\beta}(p) = \lambda_\alpha(p)g'_{\alpha\beta}(p)(\lambda_\beta(p))^{-1}$ for $p \in U_\alpha \cap U_\beta$.*

Two vector bundles over the same base space have equivalent cocycles relative to some open cover if and only if the two bundles are equivalent.

3.2. **Operations on Vector Bundles**

Many of the constructions one can do on vector spaces (direct sums, tensor products, dual space, etc) are easily transferable to vector bundles. As we could see when constructing the tangent bundle, the appropriate topology and smooth structure on the total space is usually clear, but working out the details can be quite tedious. Because of this, we will omit mentioning these in detail.

Direct Sum

We first consider the direct sum. Let $\pi_1 : E_1 \to M$ and $\pi_2 : E_2 \to M$ be two vector bundles over a smooth manifold M. We define the direct sum (also called the Whitney sum) of the vector bundles E_1 and E_2 to be the set

$$E_1 \oplus E_2 = \{(u_1, u_2) \in E_1 \times E_2 : \pi_1(u_1) = \pi_2(u_2)\}$$

with the projection $\pi : E_1 \oplus E_2 \to M$ given by

$$\pi(u_1, u_2) = \pi_1(u_1)(= \pi_2(u_2)).$$

If E_1 has rank k_1 and E_2 has rank k_2, then $E_1 \oplus E_2$ has rank $k_1 + k_2$. Suppose $\{\psi_\alpha\}$ and $\{\psi'_\alpha\}$ are trivializations for E_1 and E_2 respectively, corresponding to the trivializing open cover $\{U_\alpha\}$. The trivializations for $E_1 \oplus E_2$ are given by

$$\begin{aligned} \psi_\alpha \oplus \psi'_\alpha : \pi^{-1}(U_\alpha) \to& U_\alpha \times (\mathbb{R}^{k_1} \oplus \mathbb{R}^{k_2}) \\ (u_1, u_2) \mapsto& (\pi(u_1, u_2), \psi_\alpha(u_1), \psi'_\alpha(u_2)). \end{aligned}$$

We can also describe this bundle using the transition functions

$$\begin{pmatrix} g_{\alpha\beta} & 0 \\ 0 & g'_{\alpha\beta} \end{pmatrix},$$

where $g_{\alpha\beta}$ is a transition function for E and $g'_{\alpha\beta}$ a transition function for E'.

Pullback Bundle

Let $\pi : E \to M$ be a vector bundle. A smooth map $f : N \to M$ induces a vector bundle f^*E on N, called the pullback of E by f as follows. The total space f^*E is the subset

$$\{(p, v) \in N \times E : f(p) = \pi(v)\} \subset N \times E.$$

If $\{U_\alpha\}$ is a trivializing open cover for E, then $\{f^{-1}(U_\alpha)\}$ is a trivializing open cover for f^*E. The map $\tilde{f} : f^*E \to E$ given by $\tilde{f}(p, v) = v$ is an isomorphism of $(f^*E)_p$ onto $E_{f(p)}$ whose inverse is the inclusion $v \hookrightarrow (p, v)$.

Example 3.20. *Let M and N be manifolds of dimensions m and n respectively. Then $M \times N$ is a manifold of dimension $m + n$ (see Example 2.5) and $T(M \times N)$ is equivalent to $\pi_M^*(TM) \oplus \pi_N^*(TN)$ where $\pi_M : M \times N \to M$ and $\pi_N : M \times N \to N$ are the natural projections.*

There is an important property of pullback bundles that make the job of classifying these easier.

Theorem 3.21. *Let $\pi : E \to M$ a vector bundle and $f, g : N \to M$ be smooth maps that are smoothly homotopic (that is, there exists a smooth map $H : N \times [0,1] \to M$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$). Then f^*E and g^*E are equivalent.*

Proof. See Theorem 4.7 of Chapter 3, Section 4 in [8]. Although this proof is done in the continuous category over paracompact spaces, the form these maps have are easily seen to be smooth on manifolds (which are also paracompact spaces). □

This has a particularly useful consequence.

Corollary 3.22. *Any vector bundle over a contractible space is trivial.*

Proof. Note that any vector bundle over $\{*\}$ is trivial. Let M be a contractible space and $\pi : E \to M$ a vector bundle of rank k. Then the identity map on M, $\mathrm{id}_M : M \to M$, is homotopic to a constant map $c_p : M \to \{p\}$, $c_p(x) = p$ for all $x \in M$. It follows that $\mathrm{id}_M^* E$ is equivalent to $c_p^*(\{p\} \times \mathbb{R}^k)$. But $c_p^*(\{p\} \times \mathbb{R}^k)$ is equivalent to $\{p\} \times \mathbb{R}^k$ while $\mathrm{id}_M^* E$ is equivalent to E. The result now follows. □

From the above, it follows that any vector bundle over $\mathbb{R}^n$ is trivial.

Restrictions, Subbundles and Quotients

If N is a submanifold of M, then one can consider the restriction of the vector bundle $\pi : E \to M$ to N: we let $E|_N = \pi^{-1}(N)$ and let the projection $E|_N \to N$ to be the restriction $\pi|_{E|_N}$. This is of course a vector bundle over N in the obvious way. Note that, if $i : N \to M$ is the inclusion map, then

$$i^*E = \{(p, v) \in N \times E : p = i(p) = \pi(v)\}.$$

So $E|_N$ and i^*E are equivalent in a natural way (one that does not depend on the trivializations).

Definition 3.23. *A subbundle of a vector bundle $\pi : E \to M$ is a vector bundle $\pi' : F \to M$ such that F is a submanifold of E and, for each $p \in M$, F_p is a linear subspace of E_p.*

Let F be a subbundle of E. Since F_p is a linear subspace of E_p, we can consider the quotient space E_p/F_p above p. Gluing these together, we obtain a new bundle: the quotient bundle of E by F has as its total space the set

$$E/F = \bigcup_{p \in M} E_p/F_p,$$

and projection map $\pi' : E/F \to M$ given by $\pi'(v + F_p) = \pi(v)$, where $v + F_p \in E_p/F_p$ is the coset of $v \in E_p$. This bundle has rank $k - k'$ if E has rank k and F rank k'.

Example 3.24. *Let S be a submanifold of M. Then TS is a subbundle of $TM|_S$. The normal bundle of S in M is the quotient bundle $TM|_S/TS$. We denote this bundle by NS. The fiber at p of $TM|_S/TS$ is called the normal space at p of S in M.*

Example 3.25. *Consider the n-sphere, S^n. If $x \in S^n$ (seen as a vector in $\mathbb{R}^{n+1}$), then $x_x = (x, x) \in NS^n_x$ is a normal vector at x. Note that $\dim NS^n_x = \dim T_x\mathbb{R}^3 - \dim T_xS^2 = 3 - 2 = 1$, so any normal vector at x can be written as $\lambda x_x = (x, \lambda x)$, for $\lambda \in \mathbb{R}$. A tangent vector at x can be written as (x, v), where $\langle x, v\rangle = 0$ (here $\langle\ ,\ \rangle$ is the standard inner product on $\mathbb{R}^n$), see Figure 3.1.*

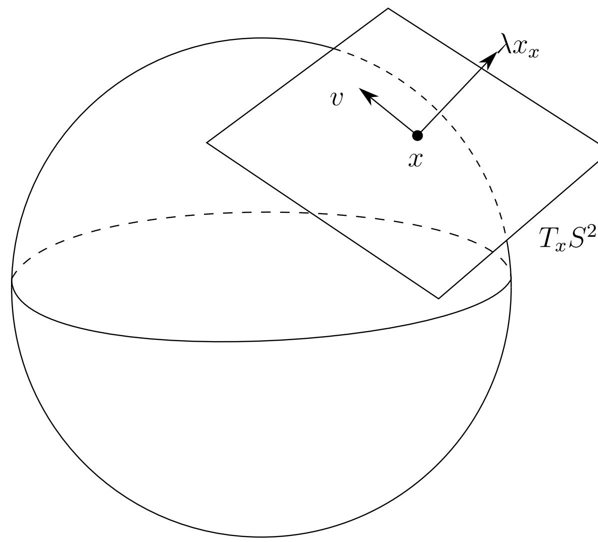

FIGURE 3.1. Tangent space at $x \in S^2$ with normal vector λx_x.

Example 3.26. *The canonical line bundle $\pi : E \to \mathbb{R}P^n$ is the subbundle of $\mathbb{R}P^n \times \mathbb{R}^{n+1}$ whose total space is the subset*

$$E = \{(l, v) \in \mathbb{R}P^n \times \mathbb{R}^{n+1} : l \in \mathbb{R}P^n,\ v \in l\}.$$

From this, we can construct the quotient bundle $(\mathbb{R}P^n \times \mathbb{R}^{n+1})/E$, called the universal quotient bundle of $\mathbb{R}P^n$.

The constructions above can also be done for $\mathbb{C}P^n$ instead. More generally, if V is a complex vector space, we can define its projectivization

$$P(V) = \{1-\text{dimensional subspaces of } V\}.$$

By following a similar procedure as $\mathbb{R}P^n$, $P(V)$ is also a smooth manifold and hence we can consider vector bundles over it. The canonical line bundle over $P(V)$ (also called the universal subbundle) is the subbundle of $P(V) \times V$ whose total space is

$$S = \{(l, v) \in P(V) \times V : v \in l\}$$

and the universal quotient bundle is the quotient $(P(V) \times V)/S$.

Similar to the case of vector spaces, the quotient, Q, of a vector bundle $\pi : E \to M$ modulo a subbundle $\pi' : S \to M$ of E is defined by the exact sequence

$$0 \longrightarrow S \longrightarrow E \longrightarrow Q \longrightarrow 0$$

(see Definition 4.9 for the definition of an exact sequence). If $E = P(V) \times V$ and S the universal subbundle, then Q is the universal quotient bundle. The exact sequence above is called the tautological exact sequence over $P(V)$.

Complexification, Realification and Tensor Product of Vector Bundles

If $\pi : E \to M$ is a complex vector bundle of rank k, each fibre can then be seen as a real vector space dimension $2k$. The transition functions of E take values in $\mathrm{GL}(n, \mathbb{C})$, but by forgetting about the complex structure, we can consider this as the space $\mathrm{GL}(2n, \mathbb{R})$. This turn $\pi : E \to M$ into a real vector bundle of rank $2k$, for which we write $\pi_{\mathbb{R}} : E_{\mathbb{R}} \to M$.

Now suppose $\pi : E \to M$ is a real vector bundle of rank k. We can construct a complex vector bundle $\pi_{\mathbb{C}} : E \otimes \mathbb{C} \to M$ of rank k as follows with total space

$$E \otimes \mathbb{C} = \bigcup_{p \in M} E_p \otimes \mathbb{C}$$

and cocycle $\{g_{\alpha\beta} \otimes \mathbb{C}\}$. In terms of matrices, $g_{\alpha\beta} \otimes \mathbb{C}$ is the $n \times n$ matrix $g_{\alpha\beta}$ where its entries are seen as complex numbers.

If $\pi : E \to M$ is a complex vector bundle, one can show that $E_{\mathbb{R}} \otimes \mathbb{C}$ is equivalent to $E \oplus \overline{E}$, where $\overline{E}$ is the vector bundle with $\overline{E}_p$ obtained by complex conjugating the elements of E_p, and the transition functions of $\overline{E}$ are those obtained by conjugating the transition functions of E.

This construction is a particular case of a more general one: the tensor product of $\pi : E \to M$ and $\pi' : E' \to M$ has total space

$$E \otimes E' = \bigcup_{p \in M} E_p \otimes E'_p$$

and cocycle $\{g_{\alpha\beta} \otimes g'_{\alpha\beta}\}$, where $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ are the cocycles of E and E' respectively.

Dual Bundles, Exterior Products and Inner Products

Let $\pi : E \to M$ be a real vector bundle of rank k. We can construct another vector bundle of the same rank by taking the fibres at p to be the dual space of E_p, which we denote by E_p^*. If $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(n, \mathbb{R})$ is a transition function for E, then $g_{\alpha\beta}(p) : \mathbb{R}^n \to \mathbb{R}^n$ induces an invertible map of dual spaces $g_{\alpha\beta}(p)^t : (\mathbb{R}^n)^* \to (\mathbb{R}^n)^*$. We let the transition for E^* be the maps $(g^t_{\alpha\beta})^{-1}$, where $(g^t_{\alpha\beta})^{-1}(p) = (g_{\alpha\beta}(p)^t)^{-1}$.

Example 3.27. *For the tangent bundle $TM \to M$, its dual is called the cotangent bundle, denoted $T^*M \to M$. If $\omega \in T^*M$, then $\omega \in T_p^*M = (T_pM)^*$ for some $p \in M$. This is a map that takes a tangent vector $v \in T_pM$ and outputs a real number $\omega(v)$. On a chart $(U, x^1, \ldots, x^n)$ of M, we have a basis $\partial/\partial x^1|_p, \ldots, \partial/\partial x^n|_p$ on T_pM for any $p \in U$. The basis in T_p^*M dual to $\partial/\partial x^i$ is denoted by $dx^1(p), \ldots, dx^n(p)$. From linear algebra, this satisfies*

$$dx^i(p)\Big(\frac{\partial}{\partial x^j}\Big|_p\Big) = \delta^i_j = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

Note that

$$dx^i(p)\Big(\frac{\partial}{\partial x^j}\Big|_p\Big) = \frac{\partial x^i}{\partial x^j}(p).$$

Just like for finite dimensional vector spaces, there is a natural equivalence between TM and $(T^*M)^*$.

Since $E_p^* = \mathrm{Hom}(E_p, \mathbb{R})$, one can generalize the above and obtain a new bundle whose fibres are $\mathrm{Hom}(E_p, E'_p)$, the linear maps from E_p to E'_p, for two bundles $\pi : E \to M$ and $\pi' : E' \to M$. We denote this bundle by $\mathrm{Hom}(E, E')$ and call this the Hom bundle of E and E'.

From linear algebra, the (algebraic) tensor product $V \otimes V'$ of two finite dimensional real vector spaces V and V' is equivalent to the space of bilinear maps $V^* \times (V')^* \to \mathbb{R}$. So the tensor product $E \otimes E'$ of two bundles $\pi : E \to M$ and $\pi' : E' \to M$ can be seen as the bundle whose fibres are bilinear maps $E_p^* \times (E_p')^* \to \mathbb{R}$. This can be generalized for the bundle $E_1 \otimes \cdots \otimes E_r$ of the vector bundles $E_1, \ldots, E_r$. That is, $E_1 \otimes \cdots \otimes E_r$ can be seen as the space whose fibres are maps $(E_1)_p^* \times \cdots \times (E_r)_p^* \to \mathbb{R}$ that are linear in each component (we call such a map multilinear). A particular subbundle of $E_1 \otimes \cdots \otimes E_r$ we are interested in is the one whose fibres are the alternating multilinear maps $\omega : (E_1)_p^* \times \cdots \times (E_r)_p^* \to \mathbb{R}$, $p \in M$. By alternating, we mean that, if σ is a permutation of $1, \ldots, r$, then

$$\omega(v_{\sigma(1)}, \ldots, v_{\sigma(r)}) = (\operatorname{sign} \sigma)\omega(v_1, \ldots, v_r).$$

For a vector space V with $\dim V = n$, we write $\bigwedge^r V$ for the space of alternating multilinear maps $V^* \times \cdots \times V^* \to \mathbb{R}$ (r-fold product). The justification for this notation is as follows. If $v_1 : V^* \to \mathbb{R}, \ldots, v_r : V^* \to \mathbb{R}$ are linear maps, one has a multilinear map $v_1 \otimes \cdots \otimes v_r : V^* \times \cdots \times V^* \to \mathbb{R}$ by

$$v_1 \otimes \cdots \otimes v_r(w_1, \ldots, w_r) = v_1(w_1) \cdots v_r(w_r).$$

We can then define the exterior product of $v_1, \ldots, v_r \in V^*$, denoted $v_1 \wedge \cdots \wedge v_r$, to be the multilinear map given by

$$v_1 \wedge \cdots \wedge v_r(w_1, \ldots, w_r) = \sum_{\sigma \in S_r} (\operatorname{sign} \sigma)(v_1 \otimes \cdots \otimes v_r)(w_{\sigma(1)}, \ldots, w_{\sigma(r)}),$$

where S_r is the symmetric group on r elements (that is, the space of all bijections $\{1, \ldots, r\} \to \{1, \ldots, r\}$). This map is alternating and so it lies in $\bigwedge^r V$. Moreover, $v_1 \wedge v_2 = -v_2 \wedge v_1$. It is common to write $\sigma \cdot (v_1 \otimes \cdots \otimes v_r)$ for the multilinear map given by

$$\sigma \cdot v_1 \otimes \cdots \otimes v_r(w_1, \ldots, w_r) = (v_1 \otimes \cdots \otimes v_r)(w_{\sigma(1)}, \ldots, w_{\sigma(r)}).$$

More generally, if ω and τ are multilinear functions from the k-fold product $V^* \times \cdots \times V^*$ and the l-fold product $V^* \times \cdots \times V^*$ into $\mathbb{R}$, respectively, then we define

$$\omega \wedge \tau = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (\operatorname{sign}(\sigma))\sigma \cdot (\omega \otimes \tau).$$

A basis for the space of multilinear maps $V^* \times \cdots \times V^* \to \mathbb{R}$ (r-fold product) is given by $e_{i_1}^* \otimes \cdots \otimes e_{i_r}^*$, $i_1, \ldots, i_r \in \{1, \ldots, n\}$, where $e_1^*, \ldots, e_n^*$ is the basis dual to the basis $e_1, \ldots, e_n$ for V. From this, one can show that a basis for $\bigwedge^r V$ is $e_{i_1} \wedge \cdots \wedge e_{i_r}$, where $1 \le i_1 < \cdots < i_r \le n$. One can also make the above discussion by using different vector spaces in the r-fold product $V \times \cdots \times V$, but we will not be dealing with this. We refer the reader to Chapter 4 of [9] for a quick treatise on multilinear maps and alternating tensors. The equivalence between multilinear maps and the algebraic tensor product can be found in Proposition 12.10 of [1] .

From the above, we can construct the vector bundle $\bigwedge^r E \to M$ from $\pi : E \to M$, whose fibres are the alternating multilinear maps $\bigwedge^r (E_p) \to M$.

Example 3.28. *Throughout this paper, we will be mainly discussing sections on the bundle $\bigwedge^r (T^*M)$. A section $\omega : M \to \bigwedge^k (T^*M)$ is called a differential k-form (or a k-form for short). For $p \in M$, $\omega(p)$ is an alternating multilinear map that takes k*

tangent vectors $v_1, \dots, v_k$ at p, and outputs a real number $\omega(p)(v_1, \dots, v_k)$. But we also have sections $v_1, \dots, v_k : U \to TM$ on coordinate neighbourhoods U of M. So we can consider the function $U \to \mathbb{R}$ given by

$$\omega(v_1, \dots, v_k)(p) = \omega(p)(v_1(p), \dots, v_k(p)).$$

We say that a k-form ω is smooth if $\omega(v_1, \dots, v_k) \in C^\infty(U)$ (this in fact agrees with the previous definition of a smooth section). On the coordinate neighbourhood U of M, we have a smooth frame $dx^1, \dots, dx^n : U \to T^*M$. This induces a smooth frame $dx^{i_1} \wedge \cdots \wedge dx^{i_k} : U \to \bigwedge^k(T^*M)$, $1 \le i_1 < \cdots < i_k \le n$ given by

$$(dx^{i_1} \wedge \cdots \wedge dx^{i_k})(p) = dx^1(p) \wedge \cdots \wedge dx^k(p).$$

One also has the smooth frame $\partial/\partial x^1, \dots, \partial/\partial x^n : U \to TM$. We then have smooth functions $U \to \mathbb{R}$ given by

$$(dx^{i_1} \wedge \cdots \wedge dx^{i_k})\Big(\frac{\partial}{\partial x^{j_1}}, \dots, \frac{\partial}{\partial x^{j_k}}\Big)(p) = (dx^{i_1}(p) \wedge \cdots \wedge dx^{i_k}(p))\Big(\frac{\partial}{\partial x^{j_1}}\Big|_p, \dots, \frac{\partial}{\partial x^{j_k}}\Big|_p\Big),$$

where $1 \le j_1, \dots, j_k \le n$.

We will discuss the space $\Gamma\big(\bigwedge^k(T^*M)\big)$ of smooth sections $M \to \bigwedge^k(T^*M)$ in the next section.

Before concluding our discussion on constructions on vector bundles, we shall briefly discuss inner products.

Definition 3.29. *A Riemannian metric (or more simply, a metric) on a vector bundle $\pi : E \to M$ is a smooth section $g : M \to E^* \otimes E^*$ such that $g(p)$ is an inner product on E_p.*

One can think of a Riemannian metric as a way to assign an inner product on each fibre E_p of E that varies smoothly on p. By varying smoothly on p, we mean that, if $v_1, \dots, v_n : U \to E$ is a smooth frame on the open subset $U \subset M$, then $g(v_i, v_j) : U \to \mathbb{R}$ given by

$$g(v_i, v_j)(p) = g(p)(v_i(p), v_j(p)),$$

for $1 \le i, j \le n$, is a smooth function.

Proposition 3.30. *Any vector bundle $\pi : E \to M$ admits a metric.*

Proof. Take a trivializing open cover $\{U_\alpha\}$ for E. Then $E|_{U_\alpha}$ is trivial so we can define a Riemannian metric $g_\alpha : U_\alpha \to E^* \otimes E^*$ (we can do this by taking a smooth frame over U_α and defining it to be orthonormal). Take a partition of unity $\{\psi_\alpha\}$ subordinate to $\{U_\alpha\}$ of E and define $g : M \to E^* \otimes E^*$ by $g = \sum_\alpha \psi_\alpha g_\alpha$. This is smooth and $g(p)$ is clearly symmetric and bilinear. Positive-definiteness follows from the fact that $\sum_\alpha \psi_\alpha(p) = 1$ and $\psi_\alpha(p) \ge 0$ for all p, so one of the $\psi_\alpha(p)$ must be positive, and g_α is positive-definite. $\square$

The existence of a Riemmanian metric on a real vector bundle E gives an equivalence between E and E^*. This equivalence need not hold if E is a complex vector bundle, since an inner product is not bilinear (see Corollary 6.28). Riemannian metrics also allows us to define the orthogonal complement, $F_p^\perp$ of a linear subspace $F_p \subset E_p$. So if F is a subbundle of E, we can define the orthogonal bundle of F in E, whose fibres are the orthogonal complements of F_p. We denote this bundle by $F^\perp$.

Just like for finite dimensional vector spaces, $F^{\perp}$ is equivalent to E/F. In particular, if N is a submanifold of M, then NS is equivalent to the orthogonal complement $TS^{\perp}$ of TS in $TM|_S$.

4. Differential Forms and de Rham Cohomology

4.1. Differential Forms on Manifolds

As promised in Example 3.28, we will now begin our studying our main space of interest: the space of differential forms. In this section give a few more results specific to this space and construct a functor from the category of differentiable manifolds to the category of commutative differential graded algebras, which will then lead to the definition of the de Rham cohomology of a manifold. From now on, all our maps will be smooth and our manifolds and vector bundles will be real, unless otherwise stated.

We first recall the definition we gave in Example 3.28.

Definition 4.1. *Let M be a smooth manifold of dimension n. For $k \geq 1$, a differential k-form ω on M is a smooth section $M \to \bigwedge^k(T^*M)$. A differential 0-form on M is a smooth function $f : M \to \mathbb{R}$. The space of k-forms on M is denoted by $\Omega^k(M)$. For $k < 0$, we define $\Omega^k(M) = 0$, and $\Omega^*(M) = \bigoplus_{q\in\mathbb{Z}} \Omega^q(M)$. For a k-form ω, we call k the degree of ω, denoted* $\deg\omega$.

If we define $\Omega^k(M) = 0$ for $k < 0$, and using the fact that $\Omega^k(M) = 0$ for $k > n = \dim M$, we can write $\Omega^*(M) = \bigoplus_{q\in\mathbb{Z}} \Omega^q(M)$.

Let $M = U \cup V$, where U and V are open. If $\omega_U : U \to \bigwedge^k(T^*U)$ and $\omega_V : V \to \bigwedge^k(T^*V)$ are k-forms on U and V such that $\omega_U(p) = \omega_V(p)$ for all $p \in U \cap V$, then there is a global k-form $\omega : M \to \bigwedge^k(T^*M)$ by Theorem 2.14. So a different way to look at differential forms on a manifold M is as a collection $\{\omega_U\}$ of differential forms on coordinate neighbourhoods U of M which agree on all the overlaps. Since k-forms on M can be seen as k-forms on open subsets which are diffeomorphic to open subsets of $\mathbb{R}^n$, a few results from the theory of differential forms on $\mathbb{R}^n$ carry over to manifolds.

We begin by discussing the operator $d : \Omega^k(M) \to \Omega^{k+1}(M)$, called the exterior derivative operator, defined locally as follows. Consider a coordinate neighbourhood $(U, x^1, \ldots, x^n)$ of M. For $f \in \Omega^0(M)$,

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i.$$

If ω is a k-form on M with $\omega = \sum_{i_1<\cdots<i_k} f_{i_1\ldots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ on U. Let us write $I = (i_1, \ldots, i_k)$, $1 \leq i_1 < \cdots < i_k \leq n$,

$$f_I = f_{i_1,\ldots,i_k}, \quad \text{and} \quad dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

so that ω can be expressed as $\sum_I f_I dx^I$ on U. We define $d\omega$ on U as

$$d\omega = \sum df_I \wedge dx^I.$$

We refer the reader to the discussion on exterior products in Section 3.2 for the definition and the basic properties on the topic. We will omit the wedge symbol at times and use concatenation instead. From this definition, the linearity of d is clear.

Proposition 4.2. *Let τ and ω be differential forms. Then*

$$d(\tau \wedge \omega) = d\tau \wedge \omega + (-1)^{\deg \tau} \tau \wedge d\omega.$$

Proof. If $f, g \in \Omega^0(M)$, then

$$\begin{aligned}
d(fg) &= \sum_{i=1}^{n} \frac{\partial(fg)}{\partial x^i} dx^i \\
&= \sum_{i=1}^{n} \left(g \cdot \frac{\partial f}{\partial x^i} + f \cdot \frac{\partial g}{\partial x^i} \right) dx^i \\
&= g \cdot \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i + f \cdot \sum_{i=1}^{n} \frac{\partial g}{\partial x^i} dx^i \\
&= g \cdot df + f \cdot dg
\end{aligned}$$

We now consider the general case. Since d is linear and $\wedge$ is bilinear, it suffices to prove the proposition on monomials $\tau = f_I dx^I$ and $\omega = g_J dx^J$. We have

$$\begin{aligned}
d(\tau \wedge \omega) &= d(f_I g_J) dx^I dx^J = g_J df_I dx^I dx^J + f_I dg_J dx^I dx^J \\
&= d\tau \wedge \omega + (-1)^{\deg \tau} f_I dx^I dg_J dx^J \\
&= d\tau \wedge \omega + (-1)^{\deg \tau} \tau \wedge d\omega,
\end{aligned}$$

concluding the proof. □

Since $\partial^2 f / \partial x^i \partial x^j = \partial^2 f / \partial x^j \partial x^i$ for a differentiable function f, and since $dx^i \wedge dx^j = -dx^j \wedge dx^i$, we have that $d^2 f = 0$. More generally,

Proposition 4.3. *$d^2 = 0$.*

Proof. It remains to consider a k-form $\omega = f_I dx^I$ on a chart $(U, x^1, \ldots, x^n)$,

$$d^2\omega = d(df_I dx^I) = (d^2 f_I) dx^I + (-1)^k df_I (d(c_1) dx^I),$$

where $c_1 \in C^\infty(U)$ is the constant function at $1 \in \mathbb{R}$. Since $d^2 f_I = 0$ and $d(c_1) = 0$, the result follows. □

Let $f : M \to N$ be a smooth map between manifolds. This induces a pullback map $f^* : C^\infty(N) \to C^\infty(M)$ by $f^*(g) = g \circ f$. Since 0-forms are just smooth functions, we can try to extend this to a pullback map on differential forms in the following manner. Let ω be a differential form on N. Then $f^*\omega$ is given by

$$(f^*\omega)(p)(v_1, ..., v_k) = \omega(f(p))(f_{*,p}(v_1), ..., f_{*,p}(v_k))$$

where $v_1, ..., v_k$ are tangent vectors at $p \in M$ (see Example 3.10 for the definition of the pushforward f_*).

Lemma 4.4. *Let $f : M^m \to N^n$ be smooth.*

1. *$f^* : \Omega^k(N) \to \Omega^k(M)$ is $\mathbb{R}$-linear,*
2. *$f^*(\omega \wedge \eta) = f^*(\omega) \wedge f^*(\eta)$,*
3. *In any smooth chart $(U, y^1, ..., y^k)$,*

$$f^*\left(\sum_{i_1 < \ldots < i_k} \omega_{i_1 \ldots i_k} dy^{i_1} \wedge \ldots dy^{i_k} \right) = \sum_{i_1 < \ldots < i_k} (\omega_{i_1 \ldots i_k} \circ f) d(y^{i_1} \circ f) \wedge \cdots \wedge d(y^{i_k} \circ f).$$

Proof. (1) is clear. To prove (2), let ω be k-form and η an l-form on N, and note that $f^*(\omega \otimes \eta) = f^*\omega \otimes f^*\eta$. Then, using (1),

$$\begin{aligned}
f^*(\omega \wedge \eta) &= f^*\left(\frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (\operatorname{sign} \sigma)\sigma \cdot (\omega \otimes \eta)\right) \\
&= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (\operatorname{sign} \sigma)\sigma \cdot f^*(\omega \otimes \eta) \\
&= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (\operatorname{sign} \sigma)\sigma \cdot \Big(f^*(\omega) \otimes f^*(\eta)\Big) \\
&= f^*(\omega) \wedge f^*(\eta).
\end{aligned}$$

For (3), it suffices to consider a monomial $\omega_{i_1 \dots i_k} dy^{i_1} \wedge \dots dy^{i_k} \in \Omega^k(M)$. Then, using (2),

$$\begin{aligned}
f^*(\omega_{i_1 \dots i_k} dy^{i_1} \wedge \cdots \wedge dy^{i_k}) &= f^*(\omega_{i_1 \dots i_k}) f^*(dy^{i_1}) \wedge \cdots \wedge f^*(dy^{i_k}) \\
&= (\omega_{i_1 \dots i_k} \circ f) f^*(dy^{i_1}) \wedge \cdots \wedge f^*(dy^{i_k}).
\end{aligned}$$

For $v = \partial/\partial x^r|_p \in T_pM$,

$$\begin{aligned}
(f^* dy^{i_j})(p)(v) &= dy^{i_j}(f(p))(f_{*,p}(v)) \\
&= dy^{i_j}(f(p)) \left(\sum_{q=1}^{n} \frac{\partial (y^q \circ f)}{\partial x^r}(p) \frac{\partial}{\partial y^q}\Big|_{f(p)} \right) \\
&= \sum_{q=1}^{n} \frac{\partial (y^q \circ f)}{\partial x^r}(p) \frac{\partial y^{i_j}}{\partial y^q}(f(p)) \\
&= \sum_{q=1}^{n} \frac{\partial (y^q \circ f)}{\partial x^r}(p) \delta_q^{i_j} \\
&= \frac{\partial (y^{i_j} \circ f)}{\partial x^r}(p),
\end{aligned}$$

while

$$\begin{aligned}
d(y^{i_j} \circ f)(p)(v) &= \sum_{q=1}^{m} \frac{\partial (y^{i_j} \circ f)}{\partial x^q}(p) dx^q(\partial/\partial x^r)|_p \\
&= \sum_{q=1}^{m} \frac{\partial (y^{i_j} \circ f)}{\partial x^q}(p) \delta_r^q \\
&= \frac{\partial (y^{i_j} \circ f)}{\partial x^r}(p).
\end{aligned}$$

Hence, $f^*(dy^{i_j}) = d(y^{i_j} \circ f)(= d(f^* y^{i_j}))$ which concludes the proof of (3). □

From the last part of the proof above, we have the following.

Corollary 4.5. *The pullback map f^* commutes with d.*

Proposition 4.6. *Let $F : M \to N$ and $G : N \to P$ be smooth maps and $\mathrm{id}_M : M \to M$ be the identity on M. Then $(G \circ F)^* = F^* \circ G^*$ and $\mathrm{id}_M^* = \mathrm{id}_{\Omega^k(M)}$.*

Proof. We only need to consider monomials. Let $\omega = f dx^{i_1} \dots dx^{i_k}$ be a k-form on P. Then

$$\begin{aligned}(G \circ F)^*(\omega) &= (f \circ G \circ F) d(x^{i_1} \circ G \circ F) \dots d(x^{i_k} \circ G \circ F),\\ F^* \circ G^*(\omega) &= F^*(f \circ G) d(x^{i_1} \circ G) \dots d(x^{i_k} \circ G)\\ &= (f \circ G \circ F) d(x^{i_1} \circ G \circ F) \dots d(x^{i_k} \circ G \circ F).\end{aligned}$$

Also,

$$\begin{aligned}\mathrm{id}_M^*(\omega) &= (f \circ \mathrm{id}_M) d(x^{i_1} \circ \mathrm{id}_M) \dots d(x^{i_k} \circ \mathrm{id}_M)\\ &= f dx^{i_1} \dots dx^{i_k},\end{aligned}$$

which is what we wanted. □

This last result shows the functoriality property of the pullback. That is, we have a mapping Ω^* that takes a differentiable manifolds M and outputs the graded algebra $\Omega^*(M)$. Moreover, if $f : M \to N$ is a smooth map, then $\Omega^*(f) = f^* : \Omega^*(N) \to \Omega^*(M)$ is a linear map of graded algebras (that is, a linear map that preserves the grading and the product structure) such that, if $g : N \to P$ is also a smooth map, then

$$\begin{aligned}\Omega^k(g \circ f) &= \Omega^k(f) \circ \Omega^k(g),\\ \Omega^k(\mathrm{id}_M) &= \mathrm{id}_{\Omega^k(M)},\end{aligned}$$

An object that behaves like Ω^* is called a contravariant functor from the category of differentiable manifolds to the category of graded algebras. If $\Omega^*(f) : \Omega^*(M) \to \Omega^*(N)$ instead, we would say that Ω^* is a covariant functor. We will continue writing f^* instead of $\Omega^*(f)$. Note that, if $f : M \to N$ is a diffeomorphism, then f^* is an isomorphism since $\mathrm{id} = (f \circ f^{-1})^* = (f^{-1})^* \circ f^*$ and $\mathrm{id} = (f^{-1} \circ f)^* = f^* \circ (f^{-1})^*$.

4.2. **The de Rham Complex**

From our discussion in the previous section, we see that the exterior derivative gives us a sequence

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \longrightarrow \dots$$

with $d^2 = 0$. Later on we will consider more instances of sequences like the above, so we might as well talk about the general theory of these to ease our life in the future. This section is the beginning of our algebraic treatment of manifolds which contrasts our very geometric discussion from Sections 1 and 2. This approach will provide us with very useful results and vocabulary.

Definition 4.7. *A direct sum of vector spaces $C = \bigoplus_{q \in \mathbb{Z}} C^q$ indexed by the integers is called a differential complex if there is a family d of linear maps $d_q : C^q \to C^{q+1}$ such that $d_{q+1} \circ d_q = 0$ for all q. The collection d of all the d_q is called the differential operator of the complex C. We define the cohomology of C to be the direct sum of* *vector spaces* $H(C) = \bigoplus_{q\in\mathbb{Z}} H^q(C)$*, where* $H^q(C) = (\ker d \cap C^q)/(\operatorname{im} d \cap C^q)$ *is called the q-th cohomology group.*

We right away see that our space $\Omega^*(M)$ with the exterior derivative operator is a differential complex, called the de Rham complex, and we can thus consider its cohomology, which will be one of our central objects of study. One can very well replace the vector spaces by R-modules, where R is a commutative ring with unity, and consider an equally interesting class of objects. We will not do this.

We now give a couple more definitions and results that will be used later.

Definition 4.8. *A map* $f : A \to B$ *between two differential complexes* (A, d_A) *and* (B, d_B) *is a chain map if* $f \circ d_A = d_B \circ f$*.*

A chain map $f : A \to B$ induces a linear map $f^* : H(A) \to H(B)$ in cohomology by $f^*([a]) = [f(a)]$ such that $(f \circ g)^* = f^* \circ g^*$ and $\mathrm{id}_A^* = \mathrm{id}_{H(A)}$. So H is a covariant functor.

Definition 4.9. *A sequence of vector spaces*

$$\dots \longrightarrow V_{i-1} \xrightarrow{f_{i-1}} V_i \xrightarrow{f_i} V_{i+1} \longrightarrow \dots$$

is exact if for all i*,* $\ker f_i = \operatorname{im} f_{i-1}$*. An exact sequence of the form*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is called a short exact sequence.

Lemma 4.10 (Five Lemma)**.** *Given a commutative diagram of vector spaces and linear maps*

$$\begin{tikzcd}
\dots \arrow[r] & A_1 \arrow[r, "f_1"] \arrow[d, "\alpha"] & A_2 \arrow[r, "f_2"] \arrow[d, "\beta"] & A_3 \arrow[r, "f_3"] \arrow[d, "\gamma"] & A_4 \arrow[r, "f_4"] \arrow[d, "\delta"] & A_5 \arrow[r] \arrow[d, "\eta"] & \dots \\
\dots \arrow[r] & B_1 \arrow[r, "g_1"'] & B_2 \arrow[r, "g_2"'] & B_3 \arrow[r, "g_3"'] & B_4 \arrow[r, "g_4"'] & B_5 \arrow[r] & \dots,
\end{tikzcd}$$

if the rows are exact, β *and* δ *are isomorphisms,* α *is onto and* η *is injective, then* γ *is also an isomorphism.*

Proof. To prove this, we use a technique called diagram chasing, which consists of following the arrows and using exactness in the obvious way to obtain the result.

Let $v \in A_3$ be such that $\gamma(v) = 0$. By commutativity, $\delta(f_3(v)) = g_3(\gamma(v)) = 0$. Since δ is an isomorphis, we have that $f_3(v) = 0$. So by exactness at A_3, we can find $a_2 \in A_2$ such that $f_2(a_2) = v$. By commutativity of the diagram, $0 = \gamma(v) = \gamma(f_2(a_2)) = g_2(\beta(a_2))$. So we can use exactness at B_2 to find some $b_1 \in B_1$ such that $g_1(b_1) = \beta(a_2)$. Since α is onto, there is some $a_1 \in A_1$ with $\alpha(a_1) = b_1$. Commutativity of the diagram implies that $\beta(f_1(a_1)) = g_1(\alpha(a_1)) = g_1(b_1) = \beta(a_2)$. Since β is an isomorphism, $f_1(a_1) = a_2$ and $0 = f_2(f_1(a_1)) = f_2(a_2) = v$, showing that γ is injective.

We now show that γ is onto. Let $v \in B_3$. Then $g_3(v) \in B_4$. Since δ is an isomorphism, there is some $a_4 \in A_4$ with $\delta(a_4) = g_3(v)$. Then

$$\eta(f_4(a_4)) = g_4(\delta(a_4)) = g_4(g_3(v)) = 0.$$

Since η is injective, we find that $f_4(a_4) = 0$. By exactness at A_4, there is some $a_3 \in A_3$ with $f_3(a_3) = a_4$. Then $g_3(\gamma(a_3)) = \delta(f_3(a_3)) = \delta(a_4) = g_3(v)$. So $\gamma(a_3) - v \in \ker g_3$

from which we can use exactness to find some $b_2 \in B_2$ such that $g_2(b_2) = \gamma(a_3) - v$. The fact that β is an isomorphism gives us an element $a_2 \in A_2$ with $\beta(a_2) = b_2$. Using the commutativity of the diagram, we find that

$$\gamma(f_2(a_2)) = g_2(\beta(a_2)) = g_2(b_2) = \gamma(a_3) - v.$$

It follows that $a_3 - f_2(a_2) \in A_3$ is such that $\gamma(a_3 - f_2(a_2)) = v$, showing that γ is onto and hence concluding the proof. $\square$

Proposition 4.11. *Let*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

be short exact sequence of differential complexes, where the maps f and g are chain maps. Then there exists a collection d^* of maps $d_q^* : H^q(C) \to H^{q+1}(A)$ such that the sequence

$$\dots \longrightarrow H^q(A) \xrightarrow{f^*} H^q(B) \xrightarrow{g^*} H^q(C) \xrightarrow{d^*} H^{q+1}(A) \longrightarrow \dots,$$

is exact.

Proof. We will only define the map d^* and omit the proof that the sequence is exact. Let $[c] \in H^q(C)$ be the cohomology class of $c \in \ker d \cap C^q$. Since the sequence

$$0 \longrightarrow A^q \xrightarrow{f_q} B^q \xrightarrow{g_q} C^q \longrightarrow 0$$

is exact for each q, we can find some $b \in B^q$ such that $g_q(b) = c$. Then $g_{q+1}(db) = d(g_q(b)) = dc = 0$, so by exactness we can find some $a \in A^{q+1}$ such that $f_{q+1}(a) = db$. Then $f_{q+2}(da) = d(f_{q+1}(a)) = d^2 b = 0$. Since f_{q+2} is injective by exactness at A^{q+2}, $da = 0$ so a defines an element $[a]$ in $H^{q+1}(A)$. We define $d^*[c] = [a]$.

Note that this is well-defined by the linearity of f_q and g_q. $\square$

We are finally ready to focus our attention on the de Rham complex previously defined.

Definition 4.12. *The differential complex $(\Omega^*(M), d)$ is called the de Rham complex of M. The cohomology of $\Omega^*(M)$ is called the de Rham cohomology of M, denoted $H^*_{dR}(M)$. A k-form ω with $d\omega = 0$ is a closed k-form. If $\omega = d\eta$ for some $(k-1)$-form η, we say that ω is exact.*

We sometimes suppress the subscript dR and we will write cohomology class of $\omega \in \ker d \cap \Omega^*(M)$ by $[\omega]$.

It is not always the case that we care about all differential forms in our space but rather a subset of these. For example, we might interested in those forms with compact support, in which case having a specific notion of the cohomology of these is needed. For a differential form ω, we define its support to be the set

$$\operatorname{supp} \omega = \operatorname{cl}_M(\{p \in M : \omega(p) \neq 0\}).$$

The exterior derivative of a form with compact support also has compact support so we can make the following definition.

Definition 4.13. *The complex $(\Omega_c^*(M), d)$, where $\Omega_c^k(M)$ is the space of k-forms on a smooth manifold M with compact support and d is the exterior derivative restricted to $\Omega_c^*(M)$, is called the de Rham complex with compact support. The cohomology, $H_c^*(M)$, of this complex is the de Rham cohomology with compact support (or the compactly supported de Rham cohomology).*

Let us compute the cohomology of some simple spaces and hopefully gain an understanding of what these equivalence classes represent.

Example 4.14 (Cohomology of a singleton)**.** *Since a singleton $\{*\}$ is a 0-dimensional manifold, $\Omega^k(\{*\}) = 0$ for $k > 0$. The smooth functions on $\{*\}$ are the constant maps $c_k : * \mapsto k \in \mathbb{R}$. Thus, $\Omega^0(\{*\}) = \mathbb{R}$. Since the only exact form is the zero map, we conclude that*

$$H^k(\{*\}) = \begin{cases} \mathbb{R}, & k = 0, \\ 0, & k > 0. \end{cases}$$

If a manifold M is such that $H^*(M) = H^*(\{*\})$, we say that M has trivial cohomology.

Example 4.15. *For a connected manifold M, we always have $H^0(M) = \mathbb{R}$. Indeed, if $f \in \Omega^0(M)$ is closed, then on a chart (U, x^i), $0 = df = \sum(\partial f/\partial x^i)dx^i$. So $\partial f/\partial x^i = 0$ for all i and so f is constant on U. Now, on any nonempty intersection of two coordinate neighbourhoods U and V, $f|_U(p) = f|_V(p)$. Thus $f|_U = f|_V$ and so f is constant on M. We conclude that $\Omega^0(M) = \mathbb{R}$ and since the zero map is the only exact form on M, we have the result for $H^0(M)$.*

Example 4.16. *If M is a connected, non-compact manifold of positive dimension, then $H_c^0(M) = 0$. This follows from the fact that the only compactly-supported constant smooth function on M is the zero map.*

This already allows us to build our intuition on what the zeroth cohomology group, $H^0(M)$, means. If M is a manifold that is not necessarily connected, then we only need to work on each connected component and pull the forms back to M through the restriction map. We then find that

$$H^0(M) = \bigoplus_{k \in I} H^0(U_k) = \bigoplus_{k \in I} \mathbb{R},$$

where $\{U_k\}_{k \in I}$ are the connected components of M. So the zeroth de Rham cohomology group counts the number of connected components in our space.

Example 4.17 (Cohomology of $\mathbb{R}$)**.** *By the previous example, $H^0(\mathbb{R}) = \mathbb{R}$, and $H^k(\mathbb{R}) = 0$ for $k > 1$. For $k = 1$, if $\omega = f dx$, then*

$$g(x) = \int_0^x f$$

is a 1-form with $dg(x) = f(x)dx$, so ω is exact and hence $H^1(\mathbb{R}) = 0$. We conclude that $\mathbb{R}$ has trivial cohomology. We will later show that this is in fact true for $\mathbb{R}^k$, a result known as the Poincaré lemma.

In the case of forms with compact support, we can consider integrating our forms. If $\omega \in \Omega_c^1(\mathbb{R})$, then $\omega = h \cdot dx$, where $h : \mathbb{R} \to \mathbb{R}$ has compact support. We define $\int_{\mathbb{R}} : \Omega_c^1(\mathbb{R}) \to \mathbb{R}$ by

$$\int_{\mathbb{R}} \omega := \int_{\mathbb{R}} h.$$

Let k be a real number and $\rho : \mathbb{R} \to \mathbb{R}$ be a smooth bump function with total integral 1. Then $\omega = k\rho\, dx$ is a compactly-supported 1-form on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} \omega = k \int_{\mathbb{R}} \rho = k.$$

Thus, $\int_{\mathbb{R}}$ is surjective. If df is an exact 1-form with compact support, then $\operatorname{supp} f$ *is contained in the interior of some interval $[a, b]$. Hence,*

$$\int_{\mathbb{R}} df = f(b) - f(a) = 0.$$

Thus, the exact forms are in the kernel of the integration map. Now if $\omega = f dx \in \Omega_c^*(\mathbb{R})$ is in the kernel of the integration map, then the function $g : \mathbb{R} \to \mathbb{R}$ given by

$$g(x) = \int_{-\infty}^{x} f$$

has compact support and $dg = \omega$. Thus, the kernel of the integration map is exactly the subspace of exact 1-forms on $\mathbb{R}$. By the first isomorphism theorem,

$$H_c^1(\mathbb{R}) = \Omega_c^1(\mathbb{R}) / \ker \int_{\mathbb{R}} = \mathbb{R}.$$

Therefore, $H_c^k(\mathbb{R}) = 0$ if $k \neq 1$, and $H_c^1(\mathbb{R}) = \mathbb{R}$.

4.3. **The Mayer-Vietoris Sequence**

Suppose we would like to compute the cohomology of a space as simple S^1 using what we have found so far. One might try and do this using the definition of de Rham cohomology, but this, although doable, is not a simple task. We know however, that S^1 can be decomposed into the union of two open sets U, V that are diffeomorphic to the real line and $U \cap V$ is the disjoint union of two real lines. But we already know how to compute the cohomology of U, V and $U \cap V$, since diffeomorphisms induce isomorphisms of differential forms. So if we could relate the cohomology of U, V and $U \cap V$ to that of S^1, then we could find $H^*(S^1)$ right away. We will work our way to achieving this. First, in the current section we will provide a general way to compute the cohomology groups of a manifold in terms of two (and hence finitely many) arbitrary open subsets, and later on work our way into "dismembering" some of our manifolds into subsets that are diffeomorphic to Euclidean space.

Suppose $M = U \cup V$ where U and V are open subsets of M. Then the diagram of inclusion maps

$$\begin{tikzcd} & U \cap V \arrow[dl, "j_U"'] \arrow[dr, "j_V"] & \\ U \arrow[dr, "i_U"'] & & V \arrow[dl, "i_V"] \\ & M & \end{tikzcd}$$

induces the diagram of restriction maps

$$\begin{tikzcd} & \Omega^*(U \cap V) & \\ \Omega^*(U) \arrow[ur, "j_U^*"] & & \Omega^*(V) \arrow[ul, "j_V^*"'] \\ & \Omega^*(M) \arrow[ul, "i_U^*"] \arrow[ur, "i_V^*"'] & \end{tikzcd}$$

where these two diagrams commute. This in turn gives us the sequence

$$0 \longrightarrow \Omega^*(M) \xrightarrow{i_U^* \oplus i_V^*} \Omega^*(U) \oplus \Omega^*(V) \xrightarrow{j_V^* - j_U^*} \Omega^*(U \cap V) \longrightarrow 0.$$

Proposition 4.18. *The sequence*

$$(1) \qquad 0 \longrightarrow \Omega^*(M) \xrightarrow{i_U^* \oplus i_V^*} \Omega^*(U) \oplus \Omega^*(V) \xrightarrow{j_V^* - j_U^*} \Omega^*(U \cap V) \longrightarrow 0.$$

is exact.

Proof. We write $i^* = i_U^* \oplus i_V^*$ and $\delta^* = j_V^* - j_U^*$.

Exactness at $\Omega^*(M)$ follows from the injectivity of the map $\Omega^*(M) \to \Omega^*(U) \oplus \Omega^*(V)$: if $\omega \in \Omega^*(M)$ is such that $\omega|_U(p) = 0$ for all $p \in U$ and $\omega|_V(p) = 0$ for all $p \in V$, then as $M = U \cup V$, we have that $\omega(p) = 0$ for all $p \in M$.

We now prove exactness at $\Omega^*(U) \oplus \Omega^*(V)$. Let $\omega \in \Omega^k(M)$. Then

$$\delta^* \circ i^*(\omega) = \delta(\omega|_U, \omega|_V) = \omega|_{U\cap V} - \omega|_{U \cap V} = 0.$$

Hence, $\operatorname{im} i^* \subset \ker \delta^*$. If $(\omega_1, \omega_2) \in \ker \delta^*$, then $\omega_1|_{U\cap V} = \omega_2|_{U \cap V}$. From this, we can define a global form $\omega \in \Omega^*(M)$ by $\omega = \omega_1$ on U and $\omega = \omega_2$ on V. It follows that

$$i^*\omega = (\omega|_U, \omega|_V) = (\omega_1, \omega_2).$$

We conclude that the sequence is exact at $\Omega^*(U) \oplus \Omega^*(V)$.

We conclude with exactness at $\Omega^*(U \cap V)$. Let $\{\rho_U, \rho_V\}$ be a partition of unity subordinate to $\{U, V\}$. Then for $\omega \in \Omega^k(U \cap V)$,

$$\delta^*(-\rho_V \omega, \rho_U \omega) = \rho_U \omega|_{U \cap V} + \rho_V \omega|_{U \cap V} = \omega.$$

This shows exactness at $\Omega^*(U \cap V)$. □

In the above proof, $\rho_V \omega$ denotes the map

$$\rho_V \omega(p) = \begin{cases} \rho_V(p)\omega(p), & p \in U \cap V, \\ 0, & p \in U \cap (M \setminus V). \end{cases}$$

Note that $\rho_V\omega$ is a smooth on U: since $(U \cap V) \cup (U \cap (M \setminus V)) = U$, so $\{U \cap V, U \cap (M \setminus V)\}$ is an open cover of U and $\rho_V\omega$ is smooth on each of those sets. Intuitively, ρ_V is a smooth function on V that slowly vanishes as one leaves V, in particular, it vanishes as one leaves $U \cap V$ in the direction of U, so this ensure that $\rho_V\omega$ vanishes smoothly as one leaves $U \cap V$.

By Proposition 4.11, the exact sequence (1) induces a long exact sequence in cohomology, called the Mayer-Vietoris sequence:
(2)

$$\dots \longrightarrow H^k(M) \longrightarrow H^k(U) \oplus H^k(V) \longrightarrow H^k(U \cap V) \xrightarrow{d^*} H^{k+1}(M) \longrightarrow \dots$$

where the maps are given by Proposition 4.11. Doing a diagram chase as in the proof of Proposition 4.11, we find that connecting homomorphism d^* is given by

$$d^*[\omega] = \begin{cases} [-d(\rho_V\omega)], & \text{on } U, \\ [d(\rho_U\omega)], & \text{on } V, \end{cases}$$

where $\{\rho_U, \rho_V\}$ is a partition of unity subordinate to the open cover $\{U, V\}$.

Example 4.19 (Cohomology of the circle)**.** *The circle has the open cover $\{U_N, U_S\}$ where U_N and U_S are diffeomorphic to $\mathbb{R}$ and $U_N \cap U_S$ has two connected components, each diffeomorphic to $\mathbb{R}$. So $H^0(U_N) = \mathbb{R} = H^0(U_S)$, $H^0(U_N \cap U_S) = \mathbb{R}^2$ and $H^k(U_N \cap U_S) = H^k(U_N) = H^k(U_S) = 0$ for $k > 0$. Since S^1 is a connected manifold, we also know that $H^0(S^1) = \mathbb{R}$. So the Mayer-Vietoris sequence for S^1 is*

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \longrightarrow H^1(S^1) \longrightarrow 0$$

By exactness, $\mathbb{R}^2 \to H^1(S^1)$ is onto, and so

$$H^1(S^1) = \mathbb{R}^2 / \ker(\mathbb{R}^2 \to H^1(S^1)) = \mathbb{R}^2/\mathrm{im}\,(\mathbb{R}^2 \to \mathbb{R}^2).$$

The map $\mathbb{R} \to \mathbb{R}^2$ is injective, and so $\dim(\mathrm{im}\,(\mathbb{R} \to \mathbb{R}^2)) = 1$. Using exactness again, we find that $\dim(\ker(\mathbb{R}^2 \to \mathbb{R}^2)) = 1$. By the rank nullity theorem, $\dim(\mathrm{im}\,(\mathbb{R}^2 \to \mathbb{R}^2)) = \dim \mathbb{R}^2 - \dim(\ker(\mathbb{R}^2 \to \mathbb{R}^2)) = 1$. Therefore, $\dim H^1(S^1) = 1$ from which we conclude that $H^1(S^1) = \mathbb{R}$.

We now wish to have a similar result for cohomology with compact support, but the pullback of a form with compact support by a smooth map need not have compact support, so we must change our strategy. Note that, if $j : U \to M$ is the inclusion map of an open subset U of M, we can define $j_* : \Omega_c^*(U) \to \Omega_c^*(M)$ to be the map that extends a form on U by zero to a form on M. Then Ω_c^* is a covariant functor that takes an open subset U of a manifold M to $\Omega_c^*(U)$ and inclusion maps $j : U \to M$ to j_*.

Let $M = U \cup V$. Then the diagram (4.3) gives rise to the commutative diagram

$$\begin{tikzcd} & \Omega_c^*(U \cap V) \arrow[dl, "(j_U)_*"'] \arrow[dr, "(j_V)_*"] & \\ \Omega^*(U) \arrow[dr, "(i_U)_*"'] & & \Omega^*(V) \arrow[dl, "(i_V)_*"] \\ & \Omega^*(M) & \end{tikzcd}$$

which then gives rise to the sequence

$$0 \longrightarrow \Omega_c^*(U\cap V) \xrightarrow{-(j_U)_*\oplus(j_V)_*} \Omega_c^*(U)\oplus\Omega_c^*(V) \xrightarrow{(i_U)_*+(i_V)_*} \Omega_c^*(M) \longrightarrow 0$$

Similar to $\Omega^k(M)$, we have the following.

Proposition 4.20. *The sequence above of forms with compact support (4.3) is exact.*

So we have an induced long exact sequence in compactly supported cohomology:
(3)

$$\ldots \longrightarrow H_c^k(U\cap V) \longrightarrow H_c^k(U)\oplus H_c^k(V) \longrightarrow H_c^k(M) \xrightarrow{d_*} H_c^{k+1}(U\cap V) \longrightarrow \ldots .$$

This is called the Mayer-Vietoris sequence for compactly supported cohomology.

4.4. **Remarks on Orientability and Integration**

Let $\{(U_\alpha,\phi_\alpha)\}$ be an atlas for a smooth manifold M. Then one has transition functions $\phi_\alpha\circ\phi_\beta^{-1} : \phi_\beta(U_\alpha\cap U_\beta)\to\phi_\alpha(U_\alpha\cap U_\beta)$. This induces a map $g_{\alpha\beta} : U_\alpha\cap U_\beta\to \mathrm{GL}(n,\mathbb{R})$ given by $g_{\alpha\beta}(p) = D(\phi_\alpha\circ\phi_\beta^{-1})(p)$ (this is in $\mathrm{GL}(n,\mathbb{R})$ since $\phi_\alpha\circ\phi_\beta^{-1}$ is a diffeomorphism).

Definition 4.21. *A manifold M is said to be orientable if there exists an atlas $\{(U_\alpha,\phi_\alpha)\}$ for M such that $\det\big(D(\phi_\alpha\circ\phi_\beta^{-1})(p)\big) > 0$ for all α,β. Such an atlas is said to be oriented. If we fix such an oriented atlas for M, we say that M is oriented.*

A diffeomorphism $f : M \to N$ between manifolds is said to be orientation-preserving if $\det\big(D(\phi_\alpha\circ f\circ\psi_\beta^{-1})\big) > 0$ and orientation-reversing if we instead have $\det\big(D(\phi_\alpha\circ f\circ\psi_\beta^{-1})\big) < 0$, for any choice of charts (U_α,ϕ_α) for N and (V_β,ψ_β) of M. So a manifold is orientable if there is an atlas whose transition functions are orientation-preserving.

One can define this more generally for vector bundles. Recall that a cocycle $\{g_{\alpha\beta}\}$ for a vector bundle is a collection of maps $g_{\alpha\beta} : U_\alpha\cap U_\beta\to\mathrm{GL}(n,\mathbb{R})$, for any trivializing open sets U_α, U_β for E (see Section 3.1).

Definition 4.22. *Let $\pi : E\to M$ be a vector bundle of rank k over M and H a subgroup of* $\mathrm{GL}(k,\mathbb{K})$*, where $\mathbb{K}=\mathbb{R}$ if E is real and $\mathbb{K}=\mathbb{C}$ is E is complex. We say that the structure group of E can be reduced to H if there is a cocycle $\{g_{\alpha\beta}\}$ such that $g_{\alpha\beta}(p)\in H$ for all α,β.*

We say that $\pi : E\to M$ is orientable if the structure group of E can be reduced to the special linear group, $\mathrm{SL}(k,\mathbb{K})$*, the space of linear transformations with positive determinant.*

Since any vector bundle E admits a Riemannian metric (see the discussion on inner products at the end of Section 3.2), we can apply the Gram-Schmidt process on the restriction of E to each trivializing open set to reduce the structure group to the orthogonal group $O(k)$ if E is real, and to the unitary group $U(k)$ if E is complex. So E is orientable if its structure group can be reduced to the special orthogonal group $SO(k)$ if E is real and to $SU(k)$ if E is complex.

Proposition 4.23. *A manifold M of dimension n is orientable if and only if there is an n-form ω such that $\omega(p) \neq 0$ for all $p \in M$.*

Proof. Consider two charts $(U, \phi) = (U, x^1, \dots, x^n)$ and $(V, \psi) = (V, y^1, \dots, y^n)$ on M. Then, on $U_\alpha \cap U_\beta$, $\partial/\partial x^i = \sum_{j=1}^n (\partial y^j/\partial x^i)\partial/\partial y^j$ by Theorem 3.7, so

$$\begin{aligned} dy^1 \wedge \cdots \wedge dy^n\Big(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\Big) &= \sum_{i_1,\dots,i_n=1}^{n} \frac{\partial y^{i_1}}{\partial x^1} \cdots \frac{\partial y^{i_n}}{\partial x^n} \\ &\quad \cdot dy^1 \wedge \cdots \wedge dy^n\Big(\frac{\partial}{\partial y^{i_1}}, \dots, \frac{\partial}{\partial y^{i_n}}\Big) \\ &= \sum_{\sigma \in S_n} (\operatorname{sign} \sigma) \prod_{j=1}^{n} \frac{\partial y^{i_j}}{\partial x^j} \cdot \delta^j_{\sigma(i_j)} \\ &= \sum_{\sigma \in S_n} (\operatorname{sign} \sigma) \prod_{j=1}^{n} \frac{\partial y^{i_j}}{\partial x^{\sigma(i_j)}}. \end{aligned}$$

Recall that $\partial y^i/\partial x^j = D_j(y^i \circ \phi^{-1}) = D_j(\psi \circ \phi^{-1})^i$ (see Section 3.1). So this last line is actually the Jacobian determinant of $\psi \circ \phi^{-1}$.

If ω is a nowhere vanishing form, then for charts $(U, \phi) = (U, x^1, \dots, x^n)$, $(V, \psi) = (V, y^1, \dots, y^n)$, $\omega = f dx^1 \wedge \cdots \wedge dx^n = g dy^1 \wedge \cdots \wedge dy^n$ on $U \cap V$, where $f : U \to \mathbb{R}$ and $g : V \to \mathbb{R}$ are non-vanishing functions. It follows that

$$f = \omega(\partial/\partial x^1, \dots, \partial/\partial x^n) = f dx^1 \wedge \cdots \wedge dx^n(\partial/\partial x^1, \dots, \partial/\partial x^n)$$

on U. If $f(p) < 0$ for all $p \in U$, then

$$\begin{aligned} 0 < -f &= \omega(-\partial/\partial x^1, \dots, \partial/\partial x^n) \\ &= g \cdot dy^1 \wedge \cdots \wedge dy^n(\partial/\partial x^1, \dots, \partial/\partial x^n) \\ &= g \cdot \det\big(D(\psi \circ \phi^{-1})\big). \end{aligned}$$

Since replacing x^1 by $-x^1$ then turns ω into $f \cdot dx^1 \wedge \cdots \wedge dx^n$, where f is now positive on U, we can assume both f and g are positive in the equality above, using the new charts. Then $\det(D(\psi \circ \phi^{-1})) > 0$ giving the result.

Suppose M is orientable and let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas with $D(\phi_\alpha \circ \phi_\beta^{-1})(p) > 0$ for all $p \in U_\alpha \cap U_\beta$ and all α, β. Consider the n-form $\eta = dx^1 \wedge \cdots \wedge dx^n$ on $\mathbb{R}^n$. From the above,

$$(\phi_\alpha \circ \phi_\beta^{-1})^*(dx^1 \wedge \cdots \wedge dx^n) = f \cdot dx^1 \wedge \cdots \wedge dx^n,$$

where $f(p) = \det\big(\phi_\alpha \circ \phi_\beta^{-1}(p)\big) > 0$ for all $p \in \mathbb{R}^n$. For each α, let ω_α be the n-form on U_α given by $\omega_\alpha = \phi_\alpha^*(dx^1 \wedge \cdots \wedge dx^n)$. Then $\omega_\alpha(p) \neq 0$ for all $p \in U_\alpha$ and on the intersection $U_\alpha \cap U_\beta$, $\omega_\beta = (f \circ \phi_\alpha)\omega_\alpha$ and $f \circ \phi_\alpha(p) > 0$ for all $p \in U_\alpha \cap U_\beta$.

Let $\{\rho_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$. Define an n-form ω on M by $\omega = \sum_\alpha \rho_\alpha \omega_\alpha$. By the above discussion and the fact that paritiions of unity are non-negative with $\rho_\alpha(p) > 0$ for at least one α, we conclude that ω is nowhere vanishing. □

Definition 4.24. *A nowhere vanishing n-form on an n-dimensional manifold is called a volume form.*

Given two volume forms ω_1 and ω_2 on a connected, orientable manifold M, the two must be related by $\omega_1 = f \cdot \omega_2$ for some $f : M \to \mathbb{R}$ with $f(p) \neq 0$ for all $p \in M$. We can define an equivalence relation on the volume forms on M by $\omega_1 \sim \omega_2$ if the function f above is positive. This defines two equivalence classes on the volume forms on M, called orientations on M. Fixing one of these defines an orientation on M (in the old sense), denoted $[M]$. The other equivalence class is sometimes written $-[M]$ and called the opposite orientation to $[M]$.

Definition 4.25. *Let M be an oriented manifold with oriented atlas $\{(U_\alpha, \phi_\alpha)\}$. Let $\{\rho_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$. For an n-form ω in M with*

$$\operatorname{supp}(\omega) = \mathrm{cl}_M(\{p \in M : \omega(p) \neq 0\})$$

compact, we define the integral of ω by

$$\int_M \omega = \sum_\alpha \int_{\mathbb{R}^n} (\phi_\alpha^{-1})^*(\rho_\alpha \omega).$$

If $\{(U_\alpha, \phi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$ are oriented atlases that define the same orientation on M, then for partitions of unity $\{\rho_\alpha\}$, $\{\chi_\beta\}$ subordinate to $\{U_\alpha\}$ and $\{U_\beta\}$, respectively, we have $\rho_\alpha \chi_\beta \omega$ has support in $U_\alpha \cap V_\beta$ and

$$\int_{U_\alpha} \rho_\alpha \chi_\beta \omega = \int_{V_\beta} \rho_\alpha \chi_\beta \omega.$$

From this, it follows that $\int_M \omega$ is independent of the choice of oriented atlases for M (as long as these define the same orientation). It is clear that $\int_M$ is linear.

Manifolds with Boundary

Let $C \subset \mathbb{R}^n$ be a curve parametrized by a smooth map $c : [0, 1] \to \mathbb{R}^n$. Then, from multivariable calculus, we know that

$$\int_C \frac{\partial f}{\partial x^1} dx^1 + \cdots + \frac{\partial f}{\partial x^n} dx^n = f(c(1)) - f(c(0))$$

for any smooth function $f \in C^\infty(\mathbb{R}^n)$. We can think of $\{c(0), c(1)\}$ as the "boundary" of C but this of course is not the topological boundary, since the topological boundary of C in $\mathbb{R}^n$ is C itself. We can consider a new notion of boundary that allows the above to hold. If we write ∂C for the boundary of C, then the equation above can be written as

$$\int_C df = \int_{\partial C} f.$$

We write $\mathbb{H}^n$ for the set

$$\mathbb{H}^n = \{(x^1, \ldots, x^n) \in \mathbb{R}^n : x^n \geq 0\},$$

and $\partial \mathbb{H}^n = \{(x^1, \ldots, x^{n-1}, 0) \in \mathbb{R}^n\}$. A map $f : U \to V$, where $U, V \subset \mathbb{H}^n$ are open, is said to be smooth if there exists open subsets $U', V' \subset \mathbb{R}^n$ with $U \subset U'$ and $V \subset V'$ and a smooth function $f' : U' \to V'$ with $f'|_U = f$. If such an f is a homeomorphism with f^{-1} also smooth, we call f a diffeomorphism.

Definition 4.26. *A smooth manifold with boundary M of dimension n is a second countable, Hausdorff space with a collection $\{(U_\alpha, \phi_\alpha)\}$, where $\{U_\alpha\}$ is an open cover of M and $\phi_\alpha : U_\alpha \to \phi(U_\alpha) \subset \mathbb{H}^n$ is a homeomorphism such that $\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$ is a diffeomorphism for each α and β. We call $\{(U_\alpha, \phi_\alpha)\}$ an atlas for M, U_α a coordinate neighbourhood, ϕ_α a coordinate map and (U_α, ϕ_α) a chart on M. We let ∂M be the set of all $p \in M$ such that there exists a coordinate map ϕ_α with $\phi_\alpha(p) \in \partial\mathbb{H}^n$, called the boundary of M.*

One can show that $M \setminus \partial M$ is a manifold of dimension n and ∂M is a manifold of dimension $n-1$ whenever $\partial M \neq \emptyset$. A manifold is a manifold with boundary, while a manifold with boundary need not be a smooth manifold.

Example 4.27. *The closed disk $D^n = \{x \in \mathbb{R}^n : (x^1)^2 + \cdots + (x^n)^2 \leq 1\}$ is a manifold with boundary. Then $\partial D^n = S^{n-1}$.*

Definition 4.28. *A closed manifold is a compact manifold without boundary.*

All our discussion about manifolds can be translated into a discussion about manifolds with boundary. In particular, an oriented manifold with boundary M of dimension n has an orientation $[M]$ induced by a nowhere vanishing n-form. If this volume form is written locally as $f \cdot dx^1 \wedge \cdots \wedge dx^n$ for a nowhere vanishing smooth function $f : M \to \mathbb{R}$, then we have an orientation on ∂M defined by the equivalence class of $(-1)^n f \cdot dx^1 \wedge \cdots \wedge dx^{n-1}$ on ∂M. We denote this induced orientation by $[\partial M]$. If $[\partial\mathbb{H}]$ is the orientation induced by the standard volume form on $\mathbb{R}^n$, then for any diffeomorphism $\phi : U \to \mathbb{H}^n$ in the oriented atlas of M, $\phi^*[\partial\mathbb{H}^n] = [\partial M]|_{U \cap \partial M}$.

We conclude this section by mentioning a very important theorem.

Theorem 4.29 (Stoke's Theorem)**.** *Let M be an oriented smooth manifold with boundary of dimension n. If ω is an $(n-1)$-form with compact support on M, then*

$$\int_M d\omega = \int_{\partial M} \omega,$$

where ∂M is given the induced orientation.

Proof. A proof of this can be found in Theorem 8.6 of [3] □

Corollary 4.30. *If M is a closed n-manifold and ω is an $(n-1)$-form on M with compact support, then*

$$\int_M d\omega = 0.$$

5. Poincaré Lemmas

5.1. Poincaré Lemma for the de Rham cohomology

We can finally begin our computation of $H^*(\mathbb{R}^n)$. Let $\pi : M \times \mathbb{R} \to M$ be the projection onto the first factor and $s : M \to M \times \mathbb{R}$ the zero section, $s(p) = (p, 0)$. In Section 3.1 we showed that M embeds diffeomorphically into $M \times \mathbb{R}$ through the zero section. So it might be tempting to find a way to relate $H^*(M)$ and $H^*(M \times \mathbb{R})$. Moreover, since $H^*(\mathbb{R})$ is trivial, we would ideally hope that $H^*(M \times \mathbb{R})$ is isomorphic to $H^*(M)$. Since $\pi \circ s = \mathrm{id}_M$, we have that the induced map $s^* \circ \pi^* = \mathrm{id}$ in cohomology. Unfortunately, finding a map $s' : M \to M \times \mathbb{R}$ with $s' \circ \pi = \mathrm{id}_{M \times \mathbb{R}}$ isn't always possible. But cohomology gives equality of closed forms up to some exact form. So we only need to find a map $s : M \to M \times \mathbb{R}$ such that $\pi^* \circ s^* = \mathrm{id} + dK$ at the level of forms, for some map $K : \Omega^*(M) \to \Omega^{*-1}(M \times \mathbb{R})$. In cohomology, we would then have that $\pi^* \circ s^* = \mathrm{id}$ and hence $H^*(M) = H^*(M \times \mathbb{R})$.

Definition 5.1. *Let $f, g : C^* \to D^*$ be chain maps of differential complexes. A family of linear maps $\{K_n : C^n \to D^{n+1} : n \in \mathbb{Z}\}$ is a chain homotopy if*

$$f_n - g_n = dK_n + K_{n+1}d.$$

In this case, we say that f and g are chain homotopic.

Our job is now to find a chain homotopy between $\pi^* \circ s^*$ and the identity, id, on $\Omega^*(M)$. Let $\omega \in \Omega^k(M \times \mathbb{R})$, $(U, x^1, \dots, x^n)$ be a coordinate system on M and $(\mathbb{R}, t)$ the standard coordinate on $\mathbb{R}$. Then every form on $M \times \mathbb{R}$ can be uniquely given locally by a linear combination of

(1) $f(x,t)dx^{i_1} \wedge \cdots \wedge dx^{i_k}$,
(2) $g(x,t)dt \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_{k-1}}$,

where, $1 \leq i_1 < \cdots < i_k \leq n$ and $f, g \in C^\infty(U \times \mathbb{R})$. Since we want to define a new form on M, we let $K : \Omega^k(M \times \mathbb{R}) \to \Omega^{k-1}(M \times \mathbb{R})$ such that K sends forms that locally look like (1) to 0 and forms that look like (2) to

$$\left(\int_0^t g(x,u)du \right) dx^{i_1} \wedge \cdots \wedge dx^{i_{k-1}}.$$

We claim that K is the desired chain homotopy. Indeed, if ω is of the form (1), then

$$\begin{aligned} K(d\omega) &= \left(\int_0^t \frac{\partial f}{\partial t}(x,u)du \right) dx^{i_1} \wedge \cdots \wedge dx^{i_k} \\ &= (f(x,t) - f(x,0))dx^{i_1} \wedge \cdots \wedge dx^{i_k}. \end{aligned}$$

Since $(\pi^* \circ s^*)\omega = (s \circ \pi)^*\omega = f(x,0)dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ and $d(K\omega) = d(0) = 0$, we conclude that

$$K(d\omega) + dK(\omega) = \omega - \pi^* \circ s^*(\omega).$$

For forms of type (2),

$$\begin{aligned} d(K\omega) &= d\left(\left(\int_0^t g(x,u)du\right)dx^{i_1}\wedge\cdots\wedge dx^{i_{k-1}}\right) \\ &= g(x,t)dt\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_{k-1}} \\ &\quad+\sum_{j=1}^n\left(\int_0^t \frac{\partial g}{\partial x^j}(x,u)du\right)dx^j\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_{k-1}}, \end{aligned}$$

while

$$\begin{aligned} K(d\omega) &= -K\Big(\sum_{j=1}^n \frac{\partial g}{\partial x^j}dt\wedge dx^j\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_{k-1}}\Big) \\ &= -\sum_{j=1}^n\left(\int_0^t \frac{\partial g}{\partial x^j}(x,u)du\right)dx^j\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_{k-1}}. \end{aligned}$$

Since $s^*\omega=0$, $\pi^*\circ s^*(\omega)=0$ and thus

$$d(K\omega)+K(d\omega)=\omega=\omega+\pi^*\circ s^*(\omega).$$

The above shows that K is a chain homotopy between the identity id on $\Omega^*(M)$ and $\pi^*\circ s^*$. This proves the following:

Proposition 5.2. *The map $s^* : H^*(M\times\mathbb{R})\to H^*(M)$ is an isomorphism with inverse $\pi^* : H^*(M)\to H^*(M\times\mathbb{R})$.*

Corollary 5.3 (Poincaré Lemma)**.**

$$H^k(\mathbb{R}^n)=\begin{cases}\mathbb{R}, & k=0\\ 0, & k>0.\end{cases}$$

Something particular to note from the proof of Proposition 5.2 is that the fact that s is the zero section is really superfluous, we could have taken s to be the section $s(p)=(p,c)$ for any $c\in\mathbb{R}$. This leads to the following,

Corollary 5.4 (Homotopy Axiom for de Rham cohomology)**.** *Smoothly homotopic maps induce the same map in cohomology.*

Proof. If H is a homotopy between the maps f and g, then $f=H\circ s_0$ and $g=H\circ s_1$, where $s_i : M\to M\times\mathbb{R}$ is given by $s_i(p)=(p,i)$. It follows that $s_0^*=(\pi^*)^{-1}=s_1^*$ and so $f^*=g^*$. □

Corollary 5.5. *Two manifolds with the same homotopy type have the same de Rham cohomology.*

Corollary 5.6. *A contractible manifold has trivial cohomology.*

Corollary 5.7. *If A is a deformation retract of M, then $H^*(A)\cong H^*(M)$.*

Up until now, it might seem that our big treatise on vector bundles in Section 3 was unnecessary since we've barely considered any of the (somewhat complicated) constructions done, but this last result justifies it. In order to study the cohomology of a manifold M, we can study vector bundles over it and obtain the same information since M embeds into the bundle.

Example 5.8 (Cohomology of Mobius Strip). *We can decompose the Mobius strip M into the union of the open sets found in Figure 2.2. Both U_1 and U_2 are diffeomorphic to $\mathbb{R}^2$ while $U_1 \cap U_2$ is a manifold with two connected components, each diffeomorphic to $\mathbb{R}^2$. Hence, $H^0(M) = H^0(U_1) = H^0(U_2) = \mathbb{R}$ and $H^0(U_1 \cap U_2) = \mathbb{R}^2$. In degree 1, $H^1(U_1) = H^1(U_2) = H^1(U_1 \cap U_2) = 0$ and the same holds for degree 2. So the Mayer-Vietoris sequence for M is*

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \longrightarrow H^1(M) \longrightarrow 0.$$

The same procedure as in Example 4.19 gives us that $H^1(M) = \mathbb{R}$. Thus, $H^0(M) = \mathbb{R} = H^1(M)$ and $H^k(M) = 0$ for $k > 1$. This could also be proven by noting that the Mobius strip retracts to a circle along the centre. So $H^*(M) = H^*(S^1)$ by the homotopy axiom.

Example 5.9 (Cohomology of the n-Sphere). *We have already considered the case $n = 1$. Consider for a moment the case $n > 1$. We can decompose the sphere into the union open sets U_N and U_S found by stereographic projection. These are diffeomorphic to $\mathbb{R}^n$ so they have trivial cohomology. The intersection $U_N \cap U_S$ is a cylinder around the equator of S^n, so this is homotopy equivalent to S^{n-1}. We prove inductively that $H^k(S^n) = \mathbb{R}$ if $k = 0, n$ and $H^k(S^n) = 0$ otherwise.*

For $n = 1$ the result was already proven in Example 4.19. Now suppose the result holds for $n \geq 1$. Then $H^0(S^{n+1}) = \mathbb{R}$ since S^{n+1} is a connected manifold. Moreover, the Mayer-Vietoris sequence tells us that

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathbb{R}^2 \longrightarrow \mathbb{R} \longrightarrow H^1(S^{n+1}) \longrightarrow 0$$

in degree 0 and 1, while in degree n,

$$0 \longrightarrow H^n(S^n) \longrightarrow H^{n+1}(S^{n+1}) \longrightarrow 0.$$

The exactness of this last sequence tells us that $\mathbb{R} = H^n(S^n) = H^{n+1}(S^{n+1})$. It remains to show that $H^1(S^{n+1}) = 0$. The map $\mathbb{R} \to H^1(S^{n+1})$ is onto, by exactness, so $\dim \mathbb{R} - \dim \ker(\mathbb{R} \to H^1(S^{n+1}) = 1 - \dim \operatorname{im}(\mathbb{R}^2 \to \mathbb{R})$. Now,

$$\dim \operatorname{im}(\mathbb{R}^2 \to \mathbb{R}) = 2 - \dim \ker(\mathbb{R}^2 \to \mathbb{R}) = 2 - \dim \operatorname{im}(\mathbb{R} \to \mathbb{R}^2) = 1,$$

since $\mathbb{R} \to \mathbb{R}^2$ is injective. We conclude that $\dim H^1(S^{n+1}) = 0$ and so $H^1(S^{n+1}) = 0$.

5.2. **Poincaré Lemma for Compactly Supported Cohomology**

One has an analogous result as the above for compactly supported forms. We will only sketch the procedure as it is similar to the above. Compactly supported forms on $M \times \mathbb{R}$ come in linear combinations of the following two types:

(1) $\pi^*\phi \cdot f(x, t)$,
(2) $\pi^*\phi \wedge f(x, t)dt$,

where ϕ is a form on M and $f : M \times \mathbb{R} \to \mathbb{R}$ is a smooth function with compact support. Since compactly supported forms don't necessarily pullback to forms with compact support, we will have to modify our chain homotopy. Define π_* on the two types of forms above by

$$\pi_*(\pi^*\phi \cdot f(x, t)) = 0,$$

and
$$\pi_*(\pi^*\phi \wedge f(x,t)dt) = \phi \cdot \int_{\mathbb{R}} f(x,t)dt,$$
and extend this linearly.

If ω is of the first type, then $d\pi_*\omega = 0$, while
$$\begin{aligned}\pi_*(d\omega) &= \pi_*((d\pi^*\phi) \cdot f(x,t)) + \pi_*(\pi^*\phi \wedge df(x,t)) = \pi_*(\pi^*\phi \wedge df(x,t)) \\ &= \phi \cdot \int_{\mathbb{R}} \frac{\partial f}{\partial t}(x,t)dt.\end{aligned}$$
Since f has compact support, f must be zero outside outside $M \times [a,b]$ so $f(x,t_1) = 0 = f(x,t_2)$ for $t_1 < a < b < t_2$. So the integral above is just $f(x,t_2) - f(x,t_1) = 0$ and so $\pi_*(d\omega) = 0$.

Now suppose ω is of the second type. Then
$$d(\pi_*\omega) = d\left(\phi \cdot \int_{\mathbb{R}} f(x,t)dt\right) = \left(\int_{\mathbb{R}} f(x,t)dt\right)d\phi + \phi \wedge \left(\sum_{j=1}^n \int_{\mathbb{R}} \frac{\partial f}{\partial x^j}(x,t)dt\right)dx^j,$$
while
$$\begin{aligned}\pi_*(d\omega) &= \pi_*\big((d\pi^*\phi) \wedge f(x,t)dt\big) + \pi_*\left(\pi^*\phi \wedge \left(\sum_{j=1}^n \frac{\partial f}{\partial x^j}dx^j \wedge dt\right)\right) \\ &= d\phi \cdot \left(\int_{\mathbb{R}} f(x,t)dt\right) + \sum_{j=1}^n \left(\int_{\mathbb{R}} \frac{\partial f}{\partial x^j}(x,t)dt\right)\phi \wedge dx^j.\end{aligned}$$
Since π_* is linear, $\pi_*d = d\pi_*$, that is, π_* is a chain map.

We have a map $\pi_* : H_c^*(M \times \mathbb{R}) \to H_c^{*-1}(M)$ since π_* above is a chain map, so we need to find a chain map $\Omega_c^*(M) \to \Omega_c^{*+1}(M \times \mathbb{R})$ and then find a chain homotopy for the composition of the two maps. For this, let e be a 1-form in $\Omega_c^1(\mathbb{R})$ with $\int_{\mathbb{R}} e = 1$ (this is just a bump form on $\mathbb{R}$). Let $e_* : \Omega_c^*(M) \to \Omega_c^{*+1}(M \times \mathbb{R})$ be given by
$$e_*(\omega) = (\pi^*\omega) \wedge e.$$
Note that $de_*(\omega) = d(\pi^*\omega) \wedge e$, while $e_*(d\omega) = \pi^*(d\omega) \wedge e = d(\pi^*\omega) \wedge e$, so $de_* = e_*d$ and so e_* defines a map $H_c^*(M) \to H_c^{*+1}(M \times \mathbb{R})$. Since e integrates to 1, $\pi_* \circ e_*(\omega) = \omega$. We produce a chain homotopy between $e_* \circ \pi_*$ and id on $\Omega_c^*(M)$. We define $K : \Omega_c^*(M \times \mathbb{R}) \to \Omega_c^{*-1}(M \times \mathbb{R})$ by
$$K(\pi^*\omega \cdot f(x,t)) = 0$$
for forms of type (1),
$$K(\pi^*\omega \wedge (f(x,t)dt)) = \omega \int_{-\infty}^t f(x,u)du - \omega \cdot \left(\int_{\mathbb{R}} f(x,t)dt\right)\left(\int_{-\infty}^t e(t)dt\right)$$
for forms of type (2), and then extend K linearly. As before, we can confirm that this is the desired chain homotopy by computing $dK - Kd$ on the forms of type (1) and (2). This gives us:

Proposition 5.10. *The map $\pi_* : H_c^*(M \times \mathbb{R}) \to H_c^{*-1}(M)$ is an isomorphism with inverse $e_* : H_c^*(M) \to H_c^{*+1}(M)$.*

Corollary 5.11 (Poincaré Lemma for Compact Support)**.**

$$H_c^k(\mathbb{R}^n) = \begin{cases} \mathbb{R}, & k = n, \\ 0, & k \neq n. \end{cases}$$

5.3. **Good Covers and the Mayer-Vietoris Argument**

So far we have computed the cohomology of manifolds that have a good cover. For example, the n-sphere has an open cover $\{U_N, U_S\}$ where U_N and U_S are diffeomorphic to $\mathbb{R}^n$. Not all manifolds are this well-behaved. For example, $\mathbb{R} \setminus \mathbb{Z}$ does not have a finite open cover whose subsets are diffeomorphic to $\mathbb{R}^n$.

Definition 5.12. *Let M be an n-manifold. An open cover $\{U_\alpha\}$ of M is a good cover if all nonempty intersections $U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}$ are diffeomorphic to $\mathbb{R}^n$. A manifold which has a finite good cover is said to be of finite type.*

Theorem 5.13. *Every manifold M has a good cover. If M is compact, we can take the cover to be finite.*

Proof. The proof of this result requires a deeper treatise on Riemannian manifolds, so we will omit this. One can work out the proof using Problem 9.32 of [3]. □

We say that a subset J of a directed set I is cofinal in I if for every i in I there is a $j \in J$ such that $i < j$. We can consider the set of all covers of a manifold as a directed set (where $\mathcal{U} < \mathcal{V}$ if the cover $\mathcal{V}$ is a refinement of $\mathcal{U}$). The above theorem shows that the collection of good covers of a manifold M is cofinal in the set of all covers of M.

Lemma 5.14 (The Mayer-Vietoris Argument)**.** *Let M be a manifold with a finite good cover $\{U_1, \ldots, U_m\}$. If $P(U)$ is a statement about open sets of M, such that*

1. *$P(U)$ is true for all open sets diffeomorphic to $\mathbb{R}^n$, and*
2. *if $P(U)$, $P(V)$ and $P(U \cap V)$ are true, then $P(U \cup V)$ is true,*

then $P(M)$ is true.

Proof. We proceed by induction on m. If $m = 1$, then $P(M) = P(U_1)$ is true by (1). Suppose now that the result holds for $m \geq 1$ and let $\{U_1, \ldots, U_{m+1}\}$ be a good cover for M satisfying the assumptions. The intersection $(U_1 \cup \cdots \cup U_m) \cap U_{m+1}$ has a good cover $\mathcal{U} = \{U_1 \cap U_{m+1}, \ldots, U_m \cap U_{m+1}\}$ of m open sets. By the induction hypothesis, one can show by induction that $P(U_1 \cup \cdots \cup U_m)$ is true, and by (1), $P(U_{m+1})$ is true. We also have

$$P((U_1 \cup \cdots \cup U_m) \cap U_{m+1}) = P((U_1 \cap U_{m+1}) \cup \cdots \cup (U_m \cap U_{m+1})),$$

which is true by inductively applying (1) and (2). Then $P(U_1 \cup \cdots \cup U_{m+1}) = P(M)$ is true. □

Proposition 5.15. *If the manifold M has a finite good cover, then its cohomology is finite dimensional. Similarly, if M has a finite good cover, then $H_c^k(M)$ is finite dimensional for all k.*

Proof. We only prove the second part of the proposition. From the Mayer-Vietoris sequence

$$\ldots \longrightarrow H_c^k(U) \oplus H_c^k(V) \xrightarrow{r} H_c^k(U \cup V) \xrightarrow{d_*} H_c^{k+1}(U \cap V) \longrightarrow \ldots$$

we conclude that if $H_c^k(U)$, $H_c^k(V)$ and $H_c^{k+1}(U \cap V)$ are finite dimensional, then so is $H_c^k(U \cup V)$ since $H_c^k(U \cup V) \cong \ker d_* \oplus \operatorname{im} d_* \cong \operatorname{im} r \oplus \operatorname{im} d_*$.

Let M be a manifold with finite good cover $\{U_1, \ldots, U_m\}$. By definition of a good cover, $H_c^k(U_i)$ and $H_c^k(U_{i_1} \cap \cdots \cap U_{i_j})$ is finite dimensional for all $i, i_1, \ldots, i_j = 1, \ldots, m$. So condition (1) of the Mayer-Vietoris argument holds. By the previous paragraph, condition (2) of the Mayer-Vietoris argument also holds and so M has finite dimensional cohomology.

The proof for the non-compact case is the analogous. □

5.4. **Künnet Formula and the Leray-Hirsch Theorem**

Recall from Section 2 that a lot of the constructions one can perform on vector spaces can also be performed on vector bundles. In Example 3.26, we considered the vector bundle over $\mathbb{R}P^n$ whose fibres over $l \in \mathbb{R}P^n$ are the points in l and then discussed the projectivization operation on a vector space. We will transfer this to vector bundles.

Let $\pi : E \to M$ be a complex vector bundle of rank k with transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(k, \mathbb{C})$. The projectivization of E is then the manifold

$$P(E) = \bigcup_{p \in M} P(E_p)$$

with the topology of the quotient of $E \setminus s_0(M)$ modulo $\sim$, where $v_1 \sim v_2$ if and only if $v_1 = \lambda v_2$ for some $\lambda \neq 0$ and $s_0 : M \to E$ is the zero section. We then have a natural map $P(\pi) : P(E) \to M$ which sends each line in $P(E_p)$ to p, i.e. $(P(\pi))^{-1}(p) = P(E_p)$, and if $U \subset M$ is a trivializing open subset of E, then $\pi^{-1}(U) \cong U \times \mathbb{C}^k$ and so $(P(\pi))^{-1}(U) \cong U \times \mathbb{C}P^{k-1}$.

This new space $P(E)$ behaves similarly to vector bundles, but the fibres are not vector spaces, instead they are other manifolds. This leads to a new type of bundle over manifolds.

Definition 5.16. *Let F be a smooth manifold. An onto smooth map $\pi : E \to M$ is a fibre bundle with fibre F if M has an open cover $\{U_\alpha\}$, called a trivializing open cover, such that there are diffeomorphisms $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ with $\phi_\alpha|_{\pi^{-1}(p)} : \pi^{-1}(p) \to \{p\} \times F$. We call E the total space, M the base space, $\pi^{-1}(p) = E_p$ the fibre at p, and the maps ϕ_α trivializations.*

Theorem 5.17 (Künneth Formula)**.** *Let M be a manifold of finite type and F another manifold. Then*

$$H^*(M \times F) \cong H^*(M) \otimes H^*(F).$$

Proof. This proof is taken from Section I.5 of [4]. We have two projection maps $\pi_M : M \times F \to M$ and $\pi_F : M \times F \to F$. We claim that $f : H^*(M) \otimes H^*(F) \to H^*(M \times F)$,

$f(\omega \otimes \eta) = \pi_M^*\omega \wedge \pi_F^*\eta$, is an isomorphism. If we define f by the formula above at the level of forms, then for ω and η two closed forms,

$$df(\omega \otimes \eta) = \pi^*(d\omega) \wedge \pi_F^*\eta + (-1)^{\deg \omega}\pi_M^*\omega \wedge \pi_F^*(d\eta) = 0.$$

So f maps closed forms to closed forms. We also see that

$$f((\omega + d\omega') \otimes (\eta + d\eta')) = \pi_M^*\omega \wedge \pi_F^*\eta + d\big(\pi_M^*\omega' \wedge \pi_F^*\eta + \pi_M^*\omega \wedge \pi_F^*\eta' + \pi_M^*\omega' \wedge \pi_F^*(d\eta')\big)$$

for ω a closed k-form on M, ω' a $(k-1)$-form on M, η a closed l-form on F and η' an $(l-1)$-form on F. Hence, f defines a map in cohomology as originally defined.

We prove the result using the Mayer-Vietoris argument. If M is diffeomorphic to $\mathbb{R}^n$, then the result is just the Poincaré lemma.

We can obtain the commutative diagram

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
\bigoplus_{p=0}^k H^p(U \cup V) \otimes H^{k-p}(F) \arrow[r, "f"] \arrow[d] & H^k((U \cup V) \times F) \arrow[d] \\
\bigoplus_{p=0}^k \Big((H^p(U) \otimes H^{k-p}(F)) \oplus (H^p(V) \otimes H^{k-p}(F))\Big) \arrow[r, "f"] \arrow[d] & H^k(U \times F) \oplus H^k(V \times F) \arrow[d] \\
\bigoplus_{p=0}^k H^p(U \cap V) \otimes H^{k-p}(F) \arrow[r, "f"] \arrow[d] & H^k((U \cap V) \times F) \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

where each column is exact (recall that tensoring an exact sequence of vector spaces with a vector space preserves exactness). If $\{U, V\}$ is a good cover, then the five-lemma implies that the result holds for $U \cup V$. By the Mayer-Vietoris argument, the result holds for any manifold of finite type. $\square$

This theorem is particularly useful for dealing with differential forms on product manifolds. But the bundle $M \times F \to M$ is a trivial bundle, so it is more fruitful to consider a generalization of this theorem.

Theorem 5.18 (Leray-Hirsch Theorem)**.** *Let $\pi : E \to M$ be a fibre bundle over M with fibre F, where M is of finite type. If there are global cohomology classes $e_1, \dots, e_r$ on E which freely generate $H^*(F)$ when restricted to F, then $H^*(E)$ is a free module over $H^*(M)$ with basis $\{e_1, \dots, e_r\}$.*

Proof. Consider the map $f : H^*(M) \otimes \mathbb{R}\{e_1, \dots, e_r\} \to H^*(E)$ given by $f(\omega \otimes e_i) = \pi^*\omega \wedge e_i$ and extend this linearly. The same argument as the Künneth formula proves the result. $\square$

6. Characteristic Classes

6.1. **Poincaré Duality**

On $\Omega^*(M)$, we have a product $\wedge$ with $\omega \wedge \eta \in \Omega^{k+l}(M)$ whenever $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M)$. We would of course hope that we can also find such a product at the cohomology level. Indeed, since $d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^k \omega \wedge d\eta$, if ω and η are closed, then $d(\omega \wedge \eta) = 0$. So the exterior product descends to cohomology. That is, we have a map $\smile\colon H^k(M) \times H^l(M) \to H^{k+l}(M)$ given by

$$([\omega], [\eta]) \mapsto [\omega] \smile [\eta] = [\omega \wedge \eta].$$

Similarly, we can define a product $\smile\colon H^k(M) \times H^l_c(M) \to H^{k+l}_c(M)$ by the same formula as above. This last product is quite useful, since it allows us to find a form of degree $n = \dim M$ with compact support, $[\omega \wedge \eta] = [\omega] \smile [\eta]$, where $[\omega] \in H^k(M)$ and $[\eta] \in H^{n-k}_c(M)$. If M is oriented we can integrate such a form, and by Stokes's theorem,

$$\begin{aligned}
\int_M (\omega + d\omega') \wedge (\eta + d\eta') &= \int_M \omega \wedge \eta + \int_M (\omega \wedge d\eta' + (d\omega') \wedge \eta + (d\omega') \wedge (d\eta')) \\
&= \int_M \omega \wedge \eta + \int_M d\big((-1)^k \omega \wedge \eta' + \omega' \wedge \eta + \omega' \wedge d\eta'\big) \\
&= \int_M \omega \wedge \eta.
\end{aligned}$$

The second line follows since ω and η are closed. This shows that $\int_M$ induces a map $\int_M : H^n(M) \to \mathbb{R}$ in cohomology. This in turn defines a bilinear map in cohomology $\int_M : H^k(M) \times H^{n-k}_c(M) \to \mathbb{R}$ given by

$$([\omega], [\eta]) \mapsto \int_M [\omega \wedge \eta].$$

Finally, we have a linear map $P : H^k(M) \to H^{n-k}_c(M)^*$ given by

$$P([\omega])([\eta]) = \int_M ([\omega] \smile [\eta]),$$

where $[\omega] \in H^k(M)$ and $[\eta] \in H^{n-k}_c(M)$.

Suppose $M = U \cup V$ for open sets U and V, each of which is diffeomorphic to $\mathbb{R}^n$. The above gives us a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H^k(U \cup V) \arrow[r] \arrow[d, "P"] & H^k(U) \oplus H^k(V) \arrow[r] \arrow[d, "P"] & H^k(U \cap V) \arrow[r, "d^*"] \arrow[d, "P"] & \cdots \\
\cdots \arrow[r] & H^{n-k}_c(U \cup V)^* \arrow[r] & H^{n-k}_c(U)^* \oplus H^{n-k}_c(V)^* \arrow[r] & H^{n-k}_c(U \cap V)^* \arrow[r, "d_*"] & \cdots
\end{tikzcd}$$

up to a factor of ± 1. By Lemma 4.10 (the Five Lemma), if we show that P is an isomorphism on $H^k(U)$, $H^k(V)$ and $H^k(U \cap V)$, then P is an isomorphism on $H^k(U \cup V)$. But that P is an isomorphism on U, V and $U \cap V$ follows right away from the Poincaré lemmas, since

$$H^k(\mathbb{R}^n) = \begin{cases} \mathbb{R}, & k = 0, \\ 0, & k \neq 0, \end{cases} \quad \text{and} \quad H^k_c(\mathbb{R}^n) = \begin{cases} \mathbb{R}, & k = n, \\ 0, & k \neq n. \end{cases}$$

So $P : H^k(U \cup V) \to H_c^{n-k}(U \cup V)^*$ is an isomorphism.

Theorem 6.1 (Poincaré Duality Theorem)**.** *Let M be a connected, oriented n-dimensional manifold of finite type. Then $P : H^k(M) \to H_c^{n-k}(M)^*$ is an isomorphism.*

Proof. Let $\{U_1, \ldots, U_m\}$ be a finite good cover. It's clear that this cover satisfies the hypothesis of the Mayer-Vietoris argument for the statement "$H^k(U)$ is isomorphic to $H_c^{n-k}(U)^*$". Hence the result holds. □

Corollary 6.2. *If M is a connected and oriented n-manifold, then $H_c^n(M) = \mathbb{R}$. If M is also compact, then $H^n(M) = \mathbb{R}$.*

It is a well-known fact that one doesn't need M to be of finite type, but a caveat in losing this assumption is that $H^k(M)$ and $H_c^{n-k}(M)$ will not necessarily have the same dimension when M is connected, since these space might now be infinite dimensional.

Corollary 6.3. *If M is connected, oriented and of finite type, then* $\dim H^k(M) = \dim H_c^{n-k}(M)$.

Definition 6.4. *The Euler characteristic of a smooth, n-dimensional manifold M, is the integer*

$$\chi(M) = \sum_{i=0}^{n} (-1)^i \dim H^i(M).$$

Corollary 6.5. *If M is a compact, orientable n-dimensional manifold, where n is odd, then $\chi(M) = 0$.*

From the above, we can see that the odd-dimensional spheres have zero Euler characteristic, same for the Möbius strip. The Euler characteristic is strongly related to the existence of nowhere vanishing vector fields on M.

We conclude this section by briefly relating $H_c^k(E)$ and $H_c^k(M)$, where $\pi : E \to M$ is a vector bundle of rank k. If E and M are orientable manifolds of finite type, the Poincaré duality gives $H_c^*(E) = (H^{n+k-*}(E))^*$. As E and M are homotopy equivalent, $(H^{n+k-*}(E))^* = (H^{n+k-*}(M))^* = H_c^{*-k}(M)$. Hence, $H_c^*(E) = H_c^{*-k}(M)$. Here we assumed that both E and M are orientable, but it is a fact that if M is orientable and $\pi : E \to M$ is an orientable vector bundle, then E is an orientable manifold.

Lemma 6.6. *Let $\pi : E \to M$ be an orientable vector bundle of rank k and M a smooth, orientable n-manifold. Then E is an orientable manifold.*

Proof. Let $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in \mathcal{A}}$ be an oriented atlas on M and $\{(V_\beta, \psi_\beta)\}_{\beta \in B}$ a trivializing open cover for E which takes values in $SL(n, \mathbb{K})$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$). Since the trivializations are $\psi_\beta : \pi^{-1}(V_\beta) \to V_\beta \times \mathbb{R}^k$, we can define a new atlas

$$\{(U_\alpha \cap V_\beta, (\phi_\alpha \times \mathrm{id}_{\mathbb{R}^k}) \circ \psi_\beta)\}_{\alpha \in A, \beta \in B}$$

for E. Then

$$\big((\phi_\alpha \times \mathrm{id}_{\mathbb{R}^k}) \circ \psi_\beta\big) \circ \big((\phi_{\alpha'} \times \mathrm{id}_{\mathbb{R}^k}) \circ (\psi_{\beta'})\big)^{-1} = (\phi_\alpha \times \mathrm{id}_{\mathbb{R}^k}) \circ (\psi_\beta \circ \psi_{\beta'}^{-1}) \circ (\phi_{\alpha'}^{-1} \times \mathrm{id}_{\mathbb{R}^k}).$$

Orientability of E is now clear. □

Proposition 6.7. *Let $\pi : E \to M$ be an oriented vector bundle over and orientable manifold M. Then $H_c^*(E) = (H_c^{*-k}(M))^*$.*

One might hope the Lemma 6.6 could be improved in a way that every vector bundle over an orientable manifold is also orientable, but this is not necessarily the case. For example, one might consider the Mobius strip a line bundle over the circle, and since the Mobius strip is a non-orientable manifold, it cannot be orientable as a vector bundle by Lemma 6.6. However, it is possible to get rid of the assumption that M is orientable in Propositon 6.7.

6.2. **The Thom Class, the Euler Class and General Classes**

We now aim to find cohomology classes that describe different geometric aspects of a compact, oriented manifold M. As hinted before, the Euler characteristic is strongly related to the existence of nowhere vanishing vector fields on orientable manifolds. This relation comes from a certain cohomology class, called the Euler class. Similarly, there are many more of these which one can study to understand better the topological and geometric aspects of a manifold.

Let $\pi : E \to M$ be an orientable vector bundle of rank k over M of finite type with dimension n. By Lemma 6.6, E is also orientable. Fix an orientation for E. Using Corollary 6.2, there exists unique elements $\mu_M \in H_c^n(M)$, $\mu_E \in H_c^{n+k}(E)$ such that, if $\eta_M \in \Omega_c^n(M)$, $\eta_E \in \Omega_c^{n+k}(E)$ are representatives of μ_M and μ_E respectively, then

$$\int_M \eta_M = 1 = \int_E \eta_E.$$

Note that any representative of μ_M and μ_E is a volume form on M and E respectively. The manifold E is also of finite type by taking the cover $\{\pi^{-1}(U_i)\}$ where $\{U_i\}$ is a good cover of M, we have two orientable manifolds E and M of finite type. By the Poincaré duality, there exists a unique class $\tau(E) \in H_c^k(E)$, called the Thom class of the vector bundle E, such that $\pi^*\mu_M \smile \tau(E) = \mu_E$. The Thom class lives in the cohomology ring of the total space E, but our interest lies in the cohomology of the base space hence, it is natural to define a new class $e(E) \in H^k(M)$, called the Euler class of the vector bundle E, by $e(E) = s^*\tau(E)$, where $s : M \to E$ is a section. Note that, if s' is another section, then $F : M \times \mathbb{R} \to E$ given by $F(p,t) = ts(p)+(1-t)s'(p)$ is a homotopy between s and s', so $s^*\tau(E) = e(E) = (s')^*\tau(E)$ is well-defined by the homotopy property of cohomology.

Proposition 6.8. *If $\overline{E}$ denotes the manifold E with the reverse orientation, then $e(\overline{E}) = -e(E)$.*

Proof. From the definition of the Euler class, $e(\overline{E}) = s^*(\tau(\overline{E}))$. Then

$$\pi^*\mu_M \smile \tau(\overline{E}) = -\mu_E = \pi^*\mu_M \smile (-\tau(E)).$$

Hence $\tau(\overline{E}) = -\tau(E)$ and the result follows. □

We have defined two cohomology classes using topological and geometric objects on manifolds, namely sections and orientations, but this does not provide an intuition of what these classes really represent. This is what we shall do next. Repeating what we just said in the last paragraph, our main focus is on the cohomology of the base space of a vector bundle, so we will mainly worry about the Euler class in this section and reference the reader to different sources for results on the Thom class whenever we use these.

As mentioned at the beginning of this section, the Euler class is strongly related to the existence of nowhere vanishing vector fields on manifolds. In fact, the Euler class obstructs the existence of these, in the sense that there is no nowhere vanishing vector field on M whenever $e(TM) \neq 0$.

Proposition 6.9. *If there is a nowhere vanishing section $s : M \to E$ of an oriented vector bundle $\pi : E \to M$ over a connected and oriented manifold M, then $e(E) = 0$.*

Proof. Fix a metric g on E. Since $s(p)$ is never zero, $|s(p)| > 0$ for all $p \in M$ where $|v|$ is the norm of $v \in TM$ induced by g. Let $\omega \in \Omega_c^{n+k}(E)$ be a representative of the Thom class of E. If $\operatorname{supp} \omega \cap s(M) = \emptyset$, then $s^*\omega = 0$ and hence, in cohomology, $e(E) = [s^*\omega] = 0$. Otherwise, as ω has compact support, $K_1 = \min\{|s(p)| : p \in M\}$ and $K_2 = \max\{|s(p)| : p \in \operatorname{supp} \omega\}$ exist and are positive. Let $K > K_2/K_1 > 0$ and define a new section $s' = K \cdot s$. This new section is nowhere vanishing and for any $p \in M$,

$$|s'(p)| = |K \cdot s(p)| = K \cdot K_1 > \frac{K_2}{K_1} \cdot K_1 = K_2.$$

This shows that $\operatorname{supp} \omega \cap s'(M) = \emptyset$ and thus $e(E) = 0$. □

Corollary 6.10. *The Euler class of a trivial bundle is zero.*

If we can find a relation between the Euler class of the tangent bundle, $e(TM)$, and the Euler characteristic of M, $\chi(M)$, then we can combine Proposition 6.9 and Corollary 6.5 to have a general statement about compact, connected, orientable manifolds of odd dimension. Such a relation between $e(TM)$ and $\chi(M)$ exists.

Theorem 6.11 (Poincaré-Hopf Theorem)**.** *For a compact, connected and oriented manifold M, $e(TM) = \chi(M) \cdot \mu_M$, where μ_M is the cohomology class of a form of degree $n = \dim M$ with total integral 1.*

We will not prove Theorem 6.11 as it requires much heavier machinery, but various proofs can be found in [3] and [4]. From this theorem,

$$\int_M e(TM) = \chi(M).$$

Corollary 6.12. *The Euler class of a compact, connected, odd-dimensional, orientable manifold is zero.*

Corollary 6.13. *There is no nowhere vanishing vector fields on even dimensional spheres.*

Proof. Recall from Example 5.9 that $H^k(S^n) = \mathbb{R}$ if $k = 0, n$ and $H^k(S^n) = 0$ otherwise. Then

$$\chi(S^{2n}) = \dim H^0(S^{2n}) + (-1)^{2n} \dim H^{2n}(S^{2n}) = 2.$$

So

$$\int_M e(TS^{2n}) = 2$$

from which it follows that $e(TS^{2n}) \neq 0$ and Proposition 6.9 gives us the result. □

We now proceed to prove a few main properties of the Euler class. Recall that a map $f : M \to N$ is proper if $f^{-1}(K)$ is compact for every compact set $K \subset N$.

Definition 6.14. *Let $f : M \to N$ be a smooth, proper map between the connected and oriented n-dimensional manifolds M and N. The degree of f is the real number* $\deg f$ *such that*

$$\int_M f^*\omega = (\deg f) \int_N \omega,$$

where $\omega \in \Omega_c^n(N)$.

The assumption that f is proper is needed in our definition, as it guarantees that the pullback of a form with compact support also has compact support. Note that the degree of f is well-defined since the map $\tilde{f}$ given by

$$\mathbb{R} \xrightarrow{(\int_N)^{-1}} H_c^n(N) \xrightarrow{f^*} H_c^n(M) \xrightarrow{\int_M} \mathbb{R}$$

is linear, so $\tilde{f}(x) = c \cdot x$. It follows that

$$\int_M f^*\omega = \tilde{f}\left(\int_N \omega\right) = c \cdot \int_N \omega.$$

Moreover, by the homotopy property of cohomology, two homotopic maps between compact, connected and oriented manifolds will have the same degree. We will not delve into the theory of degrees of maps, but we will need the following result to prove our next theorem. Recall that an orientation-preserving map $f : M \to N$ is a diffeomorphism with $\det f_{*,p} > 0$ for all $p \in M$.

Lemma 6.15. *An orientation-preserving map $f : M \to N$ between connected, oriented manifolds has positive degree.*

Proof. Suppose $T : U \to V$ is a diffeomorphism of open subsets of $\mathbb{R}^n$ and let $x^1, \dots, x^n$ and $y^1 = T^*x^1, \dots, y^n = T^*x^n$ be coordinates on $\mathbb{R}^n$. Then

$$T^*(dx^1 \wedge \cdots \wedge dx^n) = d(x^1 \circ T) \wedge \cdots \wedge d(x^n \circ T) = J(T) dy^1 \wedge \cdots \wedge dy^n,$$

where $J(T)$ is the Jacobian matrix of T (see the proof of Proposition 4.23 for reference). Since T is orientation-preserving, $J(T) > 0$. Any n-form ω on V is of the form $f dx^1 \wedge \cdots \wedge dx^n$, $f \in C^\infty(V)$, so that

$$\int_U T^*\omega = \int_U (f \circ T) J(T) dy^1 \wedge \cdots \wedge dy^n = \int_U (f \circ T) J(T).$$

Using the change of variable formula,

$$\int_U (f \circ T) J(T) = \int_{T^{-1}(V)} (f \circ T) J(T) = \int_V f \cdot (J(T)) \cdot \left|J(T^{-1})\right| = \int_V f = \int_V \omega.$$

Comparing these equalities, we obtain that $\deg T = 1$.

Generally, $\int_M f^*\omega$ is defined as the integral between open subsets of $\mathbb{R}^n$ by using partitions of unity ρ_α, pulling back $\rho_\alpha f^*\omega$ through coordinate maps and summing over α. So using the fact that f is a diffeomorphism and the above, we can conclude that $\int_M f^*\omega$ will be a sum of the partitions of unity multiplied by $\int_N \omega$, giving the result. $\square$

In particular, an orientation preserving map pullbacks μ_N to μ_M.

Lemma 6.16. *Let $\pi : E \to M$ be an oriented vector of rank k over a compact, connected, oriented manifold. Let μ_E be the orientation of E and $j_p : E_p \to E$ the inclusion map. The Thom class $\tau(E) \in H_c^k(E)$ is the unique element in $H_c^k(E)$ such that $j_p^*\tau(E) = \mu_{E_p}$.*

Proof. See Theorem 11.26 in [3]. □

Lemma 6.17 (Thom Class is Natural)**.** *Let $\pi : E \to M$ and $\rho : F \to N$ be oriented vector bundles of rank k, over oriented, compact and connected manifolds M and N. Suppose that $(\tilde{f}, f)$ is a bundle map, $\tilde{f} : E \to F$, $f : M \to N$, such that $\tilde{f}|_{E_p} : E_p \to F_{f(p)}$ is orientation-preserving. Then $\tau(E) = \tilde{f}^*(\tau(F))$.*

Proof. Write $i_p : E_p \to E$ and $j_q : F_q \to F$ the natural inclusions, $p \in M$, $q \in N$. Note that, if $v \in E_p$, then $\tilde{f}(i_p(v)) = \tilde{f}(v) = j_p(\tilde{f}(i_p(v)))$. So for any $v \in E_p$, $\tilde{f} \circ i_p = j_p \circ \tilde{f} \circ i_p$. We write μ_p for the orientation of E_p and ν_q for the orientation of F_q and $\tilde{f}_p = \tilde{f}|_{E_p}$. Using Lemma 6.16 and the fact that $\tilde{f}_p^*$ is orientation-preserving,

$$\begin{aligned} i_p^*(\tilde{f}^*(\tau(F))) &= (\tilde{f} \circ i_p)^*(\tau(F)) \\ &= (j_p \circ \tilde{f} \circ i_p)^*(\tau(F)) \\ &= (\tilde{f} \circ i_p)^* j_p^*(\tau(F)) \\ &= (\tilde{f}_p \circ i_p)^* \nu_{f(p)} \\ &= i_p^*(\tilde{f}_p^*(\nu_{f(p)})) \\ &= i_p^* \mu_p \\ &= \mu_p. \end{aligned}$$

Since p was arbitrary, $i_p^*(\tilde{f}^*(\tau(F))) = \mu_p$ for all p and so $\tilde{f}^*(\tau(F)) = \tau(E)$ by Lemma 6.16. □

Theorem 6.18 (Euler Class is Natural)**.** *Let the setting be as in Lemma 6.17. Then $f^*(e(F)) = e(E)$. In particular, $f^*(e(E)) = e(f^*E)$.*

Proof. Let $s_M : M \to E$ and $s_N : N \to F$ be the zero sections of the respective bundles. Then $\tilde{f} \circ s_M : M \to F$ and $s_N \circ f : M \to F$ are two homotopic maps (since $H(x,t) = t \cdot \tilde{f}(s_M(x)) + (1-t) \cdot s_N(f(x))$ is a smooth map) and so define the same pullback in cohomology. We conclude that

$$\begin{aligned} f^*(e(F)) = f^*(s_N^*\tau(F)) &= (s_N \circ f)^*\tau(F) \\ &= (\tilde{f} \circ s_M)^*\tau(F) \\ &= s_M^* \tilde{f}^* \tau(F) \\ &= s_M^* \tau(E) \\ &= e(E) \end{aligned}$$

as desired. □

Let $\pi_1 : E_1 \to M_1$ and $\pi_2 : E_2 \to M_2$, we can form a new vector bundle $E_1 \times E_2$ over $M_1 \times M_2$, whose fibres are $(E_1 \times E_2)_{(p,q)} = (E_1)_p \oplus (E_2)_q$. If E_1 and E_2 are oriented bundles, then $E_1 \times E_2$ also obtains an orientation. Note that, if $M_1 = M = M_2$, then

$E_1 \oplus E_2$ is equivalent to $\Delta^*(E_1 \times E_2)$, where $\Delta : M \to M \times M$ is the diagonal map $\Delta(p) = (p, p)$.

Theorem 6.19 (Whitney Sum Formula). *Let $\pi_1 : E_1 \to M_1$, $\pi_2 : E_2 \to M_2$ be oriented vector bundles of rank k_1 and k_2, respectively, over an orientable, connected, n-manifold M. Then*

$$e(E_1 \times E_2) = e(E_1) \times e(E_2)$$

and if $M_1 = M = M_2$, then

$$e(E_1 \oplus E_2) = e(E_1) \smile e(E_2).$$

Proof. Let $p_i : E_1 \times E_2 \to E_i$ be the projection maps. Then, since the orientation on each fibre of $E_1 \times E_2$ is given by concatenation of the oriented basis in the fibres of E_1 and E_2, we have that $\nu_{(p,q)} = \nu_p \times \nu_q$, where ν_x is the orientation form on $(E_1 \times E_2)_x$. From this,

$$\nu_{(p,q)} = \nu_p \times \nu_q = j_p^* \tau(E_1) \times j_q^* \tau(E_2) = j_{(p,q)}^* (\tau(E_1) \times \tau(E_2)).$$

So $\tau(E_1 \times E_2) = \tau(E_1) \times \tau(E_2)$. It follows that

$$e(E_1 \times E_2) = s_0^*(\tau(E_1) \times \tau(E_2)) = s_1^*(\tau(E_1)) \times s_2^*(\tau(E_2)) = e(E_1) \times e(E_2),$$

where $s_0 : M_1 \times M_2 \to E_1 \times E_2$, $s_1 : M_1 \to E_1$ and $s_2 : M_2 \to E_2$ are the zero sections.

Since the diagonal map $\Delta : M \to M \times M$ is a bundle equivalence,

$$e(E_1 \oplus E_2) = \Delta^* e(E_1 \times E_2) = \Delta^*(e(E_1) \times e(E_2)) = e(E_1) \smile e(E_2).$$

$\square$

Example 6.20 (Euler Characteristic of Product of Spheres). *Consider $M = S^n \times \cdots \times S^n$ (k-fold product) and $\pi_i : M \to S^n$ the projection onto the ith factor. Then $TM = \pi_1^* TS^n \oplus \cdots \oplus \pi_k^* TS^n$. By the Whitney sum formula,*

$$e(TM) = e(\pi_1^* TS^n) \smile \cdots \smile e(\pi_k^* TS^n) = \pi_1^* e(TS^n) \smile \cdots \smile \pi_k^* e(TS^n).$$

Then

$$\begin{aligned} \chi(M) &= \int_{S^n \times \cdots \times S^n} \pi_1^* e(TS^n) \smile \cdots \smile \pi_k^* e(TS^n) \\ &= \int_{S^n} e(TS^n) \cdots \int_{S^n} e(TS^n) \\ &= \left(\int_{S^n} e(TS^n) \right)^k . \end{aligned}$$

If n is odd, then $\int_{S^n} e(TS^n) = \chi(TS^n) = 0$ so $\chi(M) = 0$. On the other hand, if n is even, $\int_{S^n} e(TS^n) = 2$ so $\chi(M) = 2^k$. In particular, the Euler characteristic of the k-torus, $S^1 \times \cdots \times S^1$ (k-fold product) is zero (this could have also be deduced from Corollary 6.11).

Remark 6.21. *Let $\pi : E \to M$ be a vector bundle of rank 2 with cocycle $\{g_{\alpha\beta}\}$. One can show that, on the trivializing open set U_α, the Euler class can be described as*

$$e(E) = -\frac{1}{2\pi i} \sum_\gamma d(\rho_\gamma d \log(g_{\gamma\alpha}))$$

(see Equation 6.38 in [4]). From this, one can conclude that $e(E_1 \otimes E_2) = e(E_1) + e(E_2)$ *for a bundle* $E_1 \otimes E_2$ *which has rank 2.*

With these properties, we can state the general definition of a characteristic class.

Definition 6.22. *A characteristic class* c *of vector bundles is an assignment of a vector bundle* $E \to M$ *to a cohomology class* $c(E) \in H^*(M)$ *which is natural, in the sense that, if* $f : N \to M$ *is a smooth map, then* $f^*c(E) = c(f^*E)$.

Using categorical language, a characteristic class is a natural transformation from the functor $\mathrm{Vect}(-)$ to the cohomology functor H^*, regarded as a functor to **Set**. The functor $\mathrm{Vect}(-)$ takes a manifold M and outputs the equivalence classes $\mathrm{Vect}(M)$ of vector bundles over M, and if $f : N \to M$ is a smooth map, then $\mathrm{Vect}(f) : \mathrm{Vect}(M) \to \mathrm{Vect}(N)$ is the pullback. Note that in the definition above there is no particular mention of differential forms or de Rham cohomology (other than the fact that H^* is a functor), and that's because the definition above is valid for general cohomology theory, where H^* is a functor that satisfies a set of axioms, called the Eilenberg-Steenrod axioms for cohomology. There are many different approaches to cohomology, and hence to characteristic classes, but an important theorem in algebraic topology, the de Rham theorem, states that many of these coincide on smooth manifolds.

Something to note is that the Euler class is a characteristic class in the sense above only if f^*E is given the induced orientation. This is what makes the Euler class special: it detects the orientation of a bundle. We will later construct more characteristic classes that detect other properties of our manifolds and use these to obtain more information on its geometric and topological properties, but before that, we must return to cohomology theory.

6.3. **Chern Classes**

The Euler class gives a way to analyze oriented manifolds with oriented real vector bundles using cohomology classes. We now turn our eyes to complex vector bundles and find cohomology classes that allows us to look at the structure of the underlying base space.

Note that, since $\mathbb{C} \setminus \{0\}$ is path-connected, the group of invertible matrices $GL(k, \mathbb{C})$ is also path-connected (write $A \in GL(k, \mathbb{C})$ in its upper triangular form and connect the complex numbers in the main diagonal, which are all nonzero since A is invertible, to 1 through a path going around the unit circle). From this, if $\pi : E \to M$ is a complex vector bundle, we can connect a transition function $g_{\alpha\beta}$ to one whose determinant is a positive real number, and so the underlying real vector bundle $E_{\mathbb{R}}$ is orientable.

Consider a complex vector bundle $\pi : E \to M$ of rank k. We can consider a new space $E_0 = E \setminus s_0(M)$, where $s_0 : M \to E$ is the zero section. The restriction $\pi_0 = \pi|_{E_0} : E_0 \to M$ is a smooth surjective map. We can build a new bundle $E' \to E_0$ where the fibre over $v_p \in E_p \setminus s_0(p)$ is the quotient $E_p / \operatorname{span}(v)$. The new bundle E' now has rank $k - 1$. In [5], it is shown that we obtain an exact sequence

$$\dots \longrightarrow H^{i-2k}(M) \xrightarrow{\cup e(E)} H^i(M) \xrightarrow{\pi_0^*} H^i(E_0) \longrightarrow H^{i-2k+1}(M) \longrightarrow \dots$$

called the Gysin sequence of the bundle $E_0 \to M$, such that $H^{i-2k}(M) = 0 = H^{i-2k+1}(M)$ for $i < 2k-1$. It follows that $\pi_0^* : H^i(M) \to H^i(E_0)$ is an isomorphism. We use this to construct our next characteristic class.

Definition 6.23. *Let $\pi : E \to M$ be a complex vector bundle of rank k. The Chern classes of $\pi : E \to M$ are defined inductively by*

$$c_i(E) = \begin{cases} (\pi_0^*)^{-1} c_i(E_0), & i < k, \\ e(E_{\mathbb{R}}), & i = k, \\ 0, & i > k. \end{cases}$$

The sum

$$c(E) = 1 + c_1(E) + \cdots + c_k(E) \in H^*(M)$$

is called the total Chern class of the bundle.

The Chern classes satisfy the following properties (see [5]):

(1) (Naturality) If $\pi_1 : E_1 \to M_1$ and $\pi_2 : E_2 \to M_2$ a complex vector bundle and $(\tilde{f}, f)$ is a bundle map $\tilde{f} : E_2 \to E_1$, $f : M_2 \to M_1$, then $c_i(E_2) = \tilde{f}^* c_i(E_1)$. In particular, $f^* c(E_1) = c(f^* E_1)$.
(2) (Whitney Sum Formula) If $E_1 \to M$ and $E_2 \to M$ are vector bundles, then $c(E_1 \oplus E_2) = c(E_1) \smile c(E_2)$,
(3) if $E \to M$ is a line bundle, then $c(E) = 1 + e(E_{\mathbb{R}})$, where $e(E_{\mathbb{R}})$ is the Euler class of $E_{\mathbb{R}}$.

These properties uniquely define the Chern classes of a complex vector bundle.

Remark 6.24. *The properties above imply that the Chern classes of a trivial bundle vanish and that $c(E \oplus T) = c(E)$ when T is a trivial bundle. These properties can be proven from the definition of the Chern classes by induction, since the Euler class of a trivial bundle vanish and then an induction argument gives the result. This result is in fact needed to prove the Whitney sum formula from our definition, but there are other approaches that don't require it (see [10] for example).*

Let us go back to the projectivization of a vector bundle in order to find more properties of the Chern classes. Consider a complex vector bundle $\pi : E \to M$ of rank k and let $P(E)$ be its projectivization. Just as for vector spaces, we can find universal bundles over $P(E)$ as follows. We denote the elements of $P(E)$ by $l_p \in P(E_p)$. Since $\pi : P(E) \to M$ is a smooth map of manifolds, and $\rho : E \to M$ is a vector bundle, we have a pullback bundle $\pi^* E$ over $P(E)$. Note that $\pi^* E|_{P(E)_p} = P(E)_p \times E_p$. We can define the subbundle of $\pi^* E$ whose total space is

$$S = \{(l_p, v) \in \pi^* E : v \in l_p\},$$

called the universal subbundle over $P(E)$. Finally, we also have the quotient $Q = \pi^* E / S$.

All of this discussion can be sumarized by the diagram

$$\begin{tikzcd}
0 \arrow[r] & S \arrow[r] \arrow[dr] & \pi^*E \arrow[r] \arrow[d, "\alpha"'] & Q \arrow[r] \arrow[dl, "\beta"'] & 0 \\
 & & P(E) \arrow[drr, "\pi"] & & E \arrow[d, "\rho"] \\
 & & & & M
\end{tikzcd}$$

where the top row is exact. The fiber at l_p in S is $\{l_p\} \times \operatorname{span}(v)$, where $v \in l_p$. So the fiber at l_p in Q is E_p/l_p and hence Q has rank $k-1$, and similarly to vector spaces, $\pi^*E = S \oplus Q$. We can then obtain the projectivization $p : P(Q) \to P(E)$ and consider the pullback $\beta^*Q = S' \oplus Q'$, where S' is the universal subbundle of β^*Q and $Q' = \beta^*Q/S'$. Note that Q' has rank $k-2$ and S' is a line bundle. It follows that $\beta^*(S \oplus Q) = \beta^*(S) \oplus \beta^*(Q) = \beta^*S \oplus S' \oplus Q'$ and $\beta^*S = S$. We can repeat this procedure inductively by then taking the projectivization of $P(Q)$ and so on to have the sequence

$$\begin{tikzcd}
E \arrow[d] & \pi_1^*E = S_1 \oplus Q_1 \arrow[d] & \pi_2^*(\pi_1^*E) = S_1 \oplus S_2 \oplus Q_2 \arrow[d] & \cdots & S_1 \oplus \cdots \oplus S_{k-1} \oplus Q_{k-1} \arrow[d] \\
M & P(E) \arrow[l, "\pi_1"'] & P(Q_1) \arrow[l, "\pi_2"'] & \ldots \arrow[l, "\pi_3"'] & P(Q_{k-2}) \arrow[l, "\pi_k"']
\end{tikzcd}$$

where each S_i, $i = 1, \ldots, k-1$, and Q_{k-1} are line bundles. Moreover, if we define $\sigma_i = \pi_i \circ \cdots \circ \pi_1$, $i = 1, \ldots, k$, then

$$\sigma_i^* E = S_1 \oplus S_2 \oplus \cdots \oplus S_{i-1} \oplus Q_{i-1}.$$

We have found a manifold $F(E) = P(Q_{k-2})$ such that the pullback σ_k^*E is the direct sum of line bundles. One can further show that $\sigma_i^* : H^*(M) \to H^*(P(Q_{i-1}))$, $Q_0 = E$, is an injective map (see [5] or [4]). A space $F(E)$ with a map $\sigma_k : F(E) \to M$ like the above is called a split manifold of E. By naturality of the Chern classes and the existence of split manifolds, we can conclude the following:

Proposition 6.25 (Splitting Principle)**.** *If a polynomial identity of the Chern classes holds under the assumption that the vector bundle is a direct sum of line bundles, then it holds for all vector bundles.*

Proposition 6.26. *Let $\pi : E \to M$ be a complex vector bundle of rank k and let E^* be its dual bundle. Then $c_i(E^*) = (-1)^i c_i(E)$.*

Proof. If E is a line bundle, then $E \otimes E^* = \operatorname{Hom}(E, E)$ which has a non-trivial section by mapping p to the identity map. Then $e(E \otimes E^*) = 0$. But $E \otimes E^*$ is a complex vector bundle of rank 1, so its realification is a real vector bundle of rank 2 and so $e((E \otimes E^*)_{\mathbb{R}}) = e(E_{\mathbb{R}}) + e(E^*_{\mathbb{R}})$. Hence $c_1(E^*) = e(E^*_{\mathbb{R}}) = -e(E_{\mathbb{R}}) = -c_1(E)$.

By the splitting principle, it suffices to consider the case where E is a direct sum of line bundles $E = L_1 \oplus \cdots \oplus L_k$. Using the properties of the Chern classes:

$$c(E) = c(L_1) \smile \cdots \smile c(L_k) = (1 + c_1(L_1)) \smile \cdots \smile (1 + c_1(L_k)),$$
$$c(E^*) = c(L_1^*) \smile \cdots \smile c(L_k^*) = (1 - c_1(L_1)) \smile \cdots \smile (1 - c_1(L_k)).$$

Expanding the product and comparing to

$$c(E) = 1 + c_1(E) + \cdots + c_k(E)$$

gives the result. □

Corollary 6.27. *If $\pi : E \to M$ is a complex vector bundle of odd rank, then E and E^* are not isomorphic.*

Proof. If they were isomorphic, by naturality of the Chern class, $c_{k+1}(E^*) = c_{k+1}(E)$, but by Proposition 6.26, $c_{k+1}(E^*) = -c_{k+1}(E)$. □

We use these result to compute the Chern class of $\mathbb{C}P^n$.

Lemma 6.28. *Let $S \to \mathbb{C}P^n$ be the canonical subbundle of $\mathbb{C}P^n \times \mathbb{C}^{n+1}$ given by*

$$S = \{(l, v) \in \mathbb{C}P^n \times \mathbb{C}^{n+1} : v \in l\}$$

and $Q = (\mathbb{C}P^n \times \mathbb{C}^{n+1})/S$. Then $T\mathbb{C}P^n \cong \mathrm{Hom}(S, Q)$.

Proof. Recall that $\mathbb{C}P^n$ is the quotient of S^{2n+1} modulo the relation $x \sim -x$. At the tangent space level, looking at T_pS^{2n+1} and $T_{-p}S^{2n+1}$ as subspaces of $T\mathbb{C}^{n+1}$, the quotient identifies $(p, v) \in T_pS^{2n+1}$ with $(-p, -v)$, where $\langle p, v\rangle = 0$. So a line $l = \mathrm{span}(p) \in \mathbb{C}P^n$ determines a map $S \to Q$ by $a \cdot p \to av + \mathbb{C}^{n+1}/l$ (since v is in the orthogonal complement of l, this is nonzero if and only if a is nonzero). Conversely, such a map determines an element in $T_l\mathbb{C}P^n$ so we have the isomorphism $T_l\mathbb{C}P^n \cong \mathrm{Hom}(l, \mathbb{C}^{n+1}/l)$. Moving l around the sphere, we find that $T\mathbb{C}P^n \cong \mathrm{Hom}(S, Q)$. □

Lemma 6.29. *Let $S \to \mathbb{C}P^n$ be as above and let $T \to \mathbb{C}P^n$ be a trivial line bundle. Then $T\mathbb{C}P^n \oplus T = S^* \oplus \cdots \oplus S^*$ ($(n+1)$-fold sum).*

Proof. Since T is trivial, we have the equivalence $T \cong \mathrm{Hom}(S, S)$, since the right hand side is also trivial. It follows that $T\mathbb{C}P^n \oplus T = \mathrm{Hom}(S, Q) \oplus \mathrm{Hom}(S, S) = \mathrm{Hom}(S, Q \oplus S)$. Now $Q \oplus S$ is a trivial bundle of rank $n+1$ (the isomorphism is given by the map $(l, w, v) \mapsto (l, v + w)$, where $l \in \mathbb{C}P^n$, $v \in l$ and $w \in l^\perp$). There is then an isomorphism $Q \oplus S = T \oplus \cdots \oplus T$ ($(k+1)$-fold sum). We conclude that

$$\begin{aligned} T\mathbb{C}P^n \oplus T &\cong \mathrm{Hom}(S, T \oplus \cdots \oplus T) \\ &\cong \mathrm{Hom}(S, T) \oplus \cdots \oplus \mathrm{Hom}(S, T) \\ &\cong S^* \oplus \cdots \oplus S^*, \end{aligned}$$

as desired. □

Proposition 6.30. *The total Chern class of $T\mathbb{C}P^n \to \mathbb{C}P^n$ is $(1 + c_1(S^*))^{n+1}$.*

Proof. Using the Whitney sum formula and the previous lemma:

$$\begin{aligned} c(T\mathbb{C}P^n) &= c(S^*) \smile \cdots \smile c(S^*) \\ &= (1 + c_1(S^*)) \smile \cdots \smile (1 + c_1(S^*)) \\ &= (1 + c_1(S^*))^{n+1}. \end{aligned}$$

□

After this last computation, given a vector bundle $E \to M$, it's not hard to see that the cohomology classes $x = c_1(S^*)$ generates the cohomology of the fibres of $S \subset P(E)$, since the fibres of S are the universal subbundle of the projective space $P(E_p)$. Hence, by the Leray-Hirsch theorem,$H^*(P(E)) = H^*(M) \otimes \mathbb{R}\{1, x, \ldots, x^{k-1}\}$. It follows that

$$H^*(P(E)) = H^*(M)[x]/(x^k + c_1(E)x^{k-1} + \cdots + c_k(E)).$$

More generally, if $\sigma : F(E) \to M$ is a split manifold of E with $\sigma^* E = S_1 \oplus \cdots \oplus S_{k-1} \oplus Q_{k-1}$, then $H^*(F(E))$ has as basis all the monomials $x_1^{\alpha_1} \ldots x_{k-1}^{\alpha_{k-1}}$, where $x_i = c_1(S_i^*)$, $0 \leq \alpha_1 \leq n-1, \ldots, 0 \leq \alpha_{k-1} \leq 1$. Moreover, the map $\sigma^* : H^*(M) \to H^*(F(E))$ is injective, so it embeds $H^*(M)$. This is the first step towards the classification of vector bundles.

7. Universal Bundles and Conclusion

The projectivization of a vector bundle lead to the existence of a split manifold. More generally, instead of considering 1-dimensional subspaces of E_p, we can consider the collection of $(k-m)$-dimensional subspaces. At the vector space level, if V has dimension k, then $G_m(V)$ is the collection of all subspaces of V with codimension m. This is called the Grassmanian of V. Similar to the projective space, one finds the universal subbundle S whose fibres at each point is itself (a $k-m$ linear space), the product bundle $G_m(V) \times V$ and the quotient bundle $Q = (G_m(V) \times V)/S$. Any split manifold can be obtained from a Grassmannian applying a similar procedure to that of the split manifold. From this, any complex vector bundle $\pi : E \to M$ is the pullback of the universal quotient bundle Q over $G_k(\mathbb{C}^N)$ through a smooth map $f : M \to G_k(\mathbb{C}^N)$, for N sufficiently large. More surprisingly, there is a one-to-one correspondence between $\mathrm{Vect}_k(M)$ and $[M, G_k(\mathbb{C}^\infty)]$, where $G_k(\mathbb{C}^\infty)$ is the union of all $G_k(\mathbb{C}^n)$ over n, and $[M, G_k(\mathbb{C}^\infty)]$ are the homotopy classes of maps $M \to G_k(\mathbb{C}^\infty)$. Because of this, the Chern classes of a complex vector bundle are pullbacks of cohomology classes in a universal quotient bundle over a Grassmannian. One can find these treatments in the work in progress [Hatcher, K-theory], which is a fantastic resource on the topic. The reason why this different approach is of interest, is because it doesn't need require the smooth structure of the manifold, only the topological one. Hence, this treatment can be generalized to general topological spaces.

There are other treatments on characteristic classes using more geometric objects such as the connection and curvature of a vector bundle [11] or as obstruction as they were originally defined in [12]. The topic of characteristic classes lead to a rich collection of topics in geometry and usually prove useful in the classification of manifolds. For example, the proof of an exciting result of Milnor in 1956, [13], on the existence of 7-dimensional spheres which are homeomorphic but not diffeomorphic to the standard sphere $S^7 \subset \mathbb{R}^8$ relied on the construction of a new invariant which relied on Pontryagin classes (which are an analog to Chern classes for real vector bundles). Similarly, Calabi's conjecture, solved in [14], conjectures on the existence of certain complex manifold whose Ricci form is a representative of the first Chern class. Many other results, both solved and ongoing make use of characteristic classes and makes future work in geometry an exciting enterprise.

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